# Non-relativistic gravities and Schrödinger field theories

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## Introduction

Conformal invariance provide a good description of the critical phenomena (second order phase transition) at equilibrium. How about systems with,

- Dynamics/non-equilibrium?
- Anisotropy?

The space and time coordinates are distinct, the idea of scale transformation should be generalized to

$$(t, \mathbf{x}) \to (\lambda^z t, \lambda \mathbf{x}), \qquad z \neq 1$$

- What is the underlying geometry?
- What is the invariant field theory?

Answers at $z = 2$; *Newoton-Cartan geometry* and the *Schrödinger field theories*.

# Newton-Cartan gravity

Cartan 1923

- Galilei group is the symmetry group for Newtonian dynamics in flat space time.
- Curved spacetime formulation (arbitrary frame) of Newtonian gravity is known as Newton-Cartan gravity.
- The non-relativistic spacetime is characterized by automorphism belonging to the $Gal(d)$ group.

Methodology;

- 1st order formulation of gravity.
- Conformal method.

Application;

- Gravitational fields as external sources, conjugated to the energy density. $T^{\mu\nu} = \frac{2}{\sqrt{g}}\frac{\delta S}{\delta g_{\mu\nu}}$

## Galilei algebra and gauge fields

The centrally extended Galilei $Gal(d)$ algebra in $d+1$ dimensions:

$$[J_{ab}, J_{cd}] = 4\delta_{[a[c}\, J_{b]d]}$$

$$[J_{ab}, P_c] = 2\delta_{c[a}P_{b]}\,, \qquad [P_a, G_b] = \delta_{ab}N$$

$$[J_{ab}, G_c] = 2\delta_{c[a}G_{b]}\,, \qquad [H, G_a] = P_a$$

| $H$ | $P_a$ | $G_a$ | $J_{ab}$ | $N$ |
|---|---|---|---|---|
| $\xi^0$ | $\xi^a$ | $\Lambda^a$ | $\Lambda^{ab}$ | $\sigma$ |
| $\tau_\mu$ | ${e_\mu}^a$ | ${\omega_\mu}^a$ | ${\omega_\mu}^{ab}$ | $m_\mu$ |

# Schrödinger algebra and gauge fields

The Schrödinger algebra in $d+1$ dimensions:

$$Sch(d) = Gal(d) \oplus Dilataion(D) \oplus SCT(K)$$

| $D$ | $K$ |
|---|---|
| $\Lambda_D$ | $\Lambda_K$ |
| $b_\mu$ | $f_\mu$ |

$$[D, P_a] = -P_a\,, \qquad [D, H] = -2H\,, \qquad [H, K] = -D\,,$$

$$[D, G_a] = G_a\,, \qquad [D, K] = 2K\,, \qquad [K, P_a] = -G_a\,.$$

Hagen, Niedere, Jackiw-Pi, Gauntlett-Gomis-Townsend, Duval-Horvaíthy, Henkel, ...

## Gauge transformation

The gauge transformation of the gauge field $A_\mu$ under $\epsilon$,

$$\delta A_\mu = \partial_\mu \epsilon + [A_\mu, \epsilon] \,.$$

Their standard transformation rules for independent fields;

$$\begin{aligned}
\delta\tau_\mu &= 2\Lambda_D \tau_\mu \,, \\
\delta e_\mu{}^a &= \Lambda^a{}_b e_\mu{}^b + \Lambda^a \tau_\mu + \Lambda_D e_\mu{}^a \,, \\
\delta m_\mu &= \partial_\mu \sigma + \Lambda^a e_{\mu a} \\
\delta b_0 &= \partial_0 \Lambda_D - \Lambda^a b_a - 2\Lambda_D b_0 + \Lambda_K \,.
\end{aligned}$$

where $b_0 = \tau^\mu b_\mu$.

## Conformal (Stükelberg) method

Einstein-Hilbert Lagrangian — Poincaré invariant — in $D$ dimensions can be related to the conformal field theory of a real scalar field with weight $w_\phi = -\frac{1}{2}(D-2)$ ;

$$e^{-1}\mathcal{L}_{\mathsf{P}} = \mathcal{R} \;\longrightarrow\; \tfrac{1}{2}\phi\Box^{\mathsf{C}}\phi$$

$$(g_{\mu\nu})^{\mathsf{P}} = \varphi^2\,(g_{\mu\nu})^{\mathsf{C}}\,, \qquad (g_{\mu\nu})^{\mathsf{C}} = \eta_{\mu\nu}\,, \qquad \varphi = \phi^{\frac{2}{D-2}}$$

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This relation also works the other way around;

Scalar conformal field theories $\longrightarrow$ Poincaré invariants
Scalar Schrödinger field theories $\longrightarrow$ Galilean invariants

What are these scalar Schrödinger field theories?

## Non-relativistic conformal method

The gauge fixed theory should be invariant under dilatation $D$ and the central charge symmetry $N$. Two scalars;

$$(\tau_\mu)^{\mathsf{G}} = \varphi^2 (\tau_\mu)^{\mathsf{Sch}}\,, \quad (e_\mu{}^a)^{\mathsf{G}} = \varphi (e_\mu{}^a)^{\mathsf{Sch}}\,, \quad M_\mu = m_\mu - \tfrac{1}{\mathrm{M}} \partial_\mu \chi\,.$$

They form a complex scalar,

$$\Psi(t, \mathbf{x}) = \varphi\, e^{i\chi} \qquad \text{with} \qquad \delta\varphi = w \Lambda_D \varphi\,, \qquad \delta\chi = \mathrm{M}\,\sigma\,.$$

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The canonical form of the invariant action under Schrödinger, in the rigid case, ${e_\mu}^a = {\delta_\mu}^a$, $\tau_\mu = {\delta_\mu}^0$ and $m_\mu = 0$, is

$$S^{(n)}_{\mathrm{Sch}} = \int dt d^d\mathbf{x}\, \Psi^\star\Big(i\partial_0 - \tfrac{1}{2\mathrm{M}}\partial_a^2\Big)^n\Psi\,, \qquad w_\Psi = -d/2 + n - 1$$

## Gauging the symmetries

Under dilatation and central charge transformation,

$$\delta\Psi = (w\Lambda_D + i\mathrm{M}\sigma)\Psi$$

The covariant derivatives are naturally defined as,

$$D_0\Psi = \tau^\mu(\partial_\mu - w\, b_\mu - i\mathrm{M}\, m_\mu)\Psi\,,$$
$$D_a\Psi = e^\mu{}_a(\partial_\mu - w\, b_\mu - i\mathrm{M}\, m_\mu)\Psi\,.$$

The variation of these covariant derivatives,

$$\delta\triangle\Psi = [(w-2)\Lambda_D + i\mathrm{M}\sigma]\triangle\Psi - i\mathrm{M}(2\Lambda^a D_a\Psi - d\Lambda_K\Psi)$$
$$\delta D_0\Psi = [(w-2)\Lambda_D + i\mathrm{M}\sigma]D_0\Psi - \Lambda^a D_a\Psi - w\Lambda_K\Psi$$

## Invariance

$$\delta\Box_{\rm Sch}\Psi = -i\left(w+\tfrac{d}{2}\right)\Lambda_K\Psi\,,$$
$$\delta\Box^2_{\rm Sch}\Psi = -i\left(2w-2+d\right)\Lambda_K\Box_{\rm Sch}\Psi\,.$$

The invariance under Schrödinger symmetries, fixes the weight of $\Psi$ to $w=-\frac{d}{2}$ and $w=-\frac{d-2}{2}$. Where

$$\Box_{\rm Sch} = iD_0 - \frac{1}{2\mathrm{M}}\triangle \quad \text{and} \quad \triangle = D^a D_a\,.$$

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Real case ($\delta\phi = w\Lambda_D\phi$);

$$\delta D_0^2\phi \;\; = -2\Lambda^a D_0 D_a\phi - 2(w-1)\Lambda_K D_0\phi\,.$$

## Schrödinger Kinetic terms

Schrödinger invariant complex scalar theories with two time derivatives, $w = -\frac{d-2}{2}$.

$$\textbf{I}. \qquad \int dt d^d\mathbf{x}\, e\, \Psi^\star \Big(iD_0 - \tfrac{1}{2M}\triangle\Big)^2 \Psi\,,$$

$$\textbf{II}. \qquad \int dt d^d\mathbf{x}\, e \left| iD_0\Psi + \frac{1}{\mathrm{M}d}\left(w\triangle\Psi - \Psi^{-1}D_a\Psi D_a\Psi\right)\right|^2.$$

After gauge-fixing $\Psi = 1$, they lead to the Hořava-Lifshtz Kinetic terms;

$$S = \frac{1}{\kappa^2}\int dt d^d\mathbf{x}\, e\left(K_{ij}K^{ij} - \lambda K^2 + \mathcal{V}\right)$$

$K_{ij}(\tau, e, m)$ becomes the Galilean boost gauge field in the boost invarinat frame.

## Potential terms

Terms Made of only spatial derivative and rotation curvature are also independently invariant under Schrödinger symmetries,

$$D_a\varphi D_a\varphi\,, \qquad \varphi\triangle\varphi\,, \qquad \varphi^2 R(J)\,, \qquad D_a\varphi D_b\varphi R^{ab}(J)\,,$$
$$\varphi D_a D_b\varphi R^{ab}(J), \quad \varphi^2 R_{ab}(J)R_{ab}(J)\,, \quad \varphi^2 R_{abcd}(J)R_{abcd}(J)\,.$$

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After the gauge fixing $\varphi = 1$,

$$\begin{aligned}
&b\cdot b\,,\quad \mathcal{R}(J)\,, && 2-\text{derivative}\,,\\
&(b\cdot b)^2\,,\quad (\mathcal{D}\cdot b)^2\,,\quad (\mathcal{D}\cdot b)b\cdot b\,, && 4-\text{derivative}\,,\\
&\mathcal{R}(J)^2\,,\quad \mathcal{R}(J)b\cdot b\,,\quad \mathcal{R}(J)\mathcal{D}\cdot b\,, && 4-\text{derivative}\,,\\
&\mathcal{R}^2_{ab}(J)\,,\quad \mathcal{R}_{ab}(J)b^a b^b\,,\quad \mathcal{R}_{ab}(J)\mathcal{D}^a b^b\,, && 4-\text{derivative}\,,\\
&\mathcal{R}^2_{abcd}(J)\,, && 4-\text{derivative}\,.
\end{aligned}$$

## Newton-Cartan equations of motions

When $w_\phi = 1$, the constraint $D_0^2\phi = 0$ is invariant under Schrödinger symmetries if $D_a\phi = 0$. Relax this condition by adding the pseudo scalar field $\chi$ transforming $\delta\chi = \mathrm{M}\sigma$,

$$D_0^2\phi - \frac{2}{\mathrm{M}}(D_0 D_a\phi) D_a\chi + \frac{1}{\mathrm{M}^2}(D_a D_b\phi) D_a\chi D_b\chi = 0\,.$$

Imposing the gauge-fixing condition $\phi = 1$ and $\chi = 0$,

$$f_0 - \omega_{0a} b^a - 2M_a D_0 b^a - M_a M_b D_a b_b = 0\,.$$

## Summary

We have constructed the most general Schrödinger invariant complex/real scalar field theory with at most two time, and four spatial derivatives leading to the most general Galilean invariant 2nd order in time theory. We showed how the Hořava-Lifshitz gravity at $z = 2$ and the Newton-Cartan gravity can be embedded into this construction.

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Thank you.