Introduction

Quantum mechanics and then quantum field theory are the most successful inventions of the human mind in its search for understanding the workings of nature. Applications are made at all scales, ranging from vast galactic reaches to tiny subnuclear distances.

The successes of quantum mechanics did not come in a single step; rather new proposals arose when a developing theory became challenged by theoretical and experimental facts. Of course the variety of proposed ideas is practically unlimited, given the unlimited number of issues that are addressed. But a common thread of symmetry runs through them all.

What is symmetry? Transformations that leave unchanged the transformed entity are said to be symmetries of that entity. They are important for practical reasons: if one realization of the entity is known, then all others related by a symmetry transformation are equally accessible. But perhaps even more important is the esthetic reason: the human mind

and eye prefer symmetric configurations.

Within physics and mathematics of field theory there are specific developments that carry the above ideas to definite theorems. Since I shall speak about some of these later in my lectures, I now discuss them briefly.

Consider a dynamical system, governed by a field variable $(\Phi(t, \mathbf{r}))$**, moving in time and space and satisfying equations of motion that are derived from Hamilton's variational Action Principle. A symmetry is present if the Action is invariant against field transformations** $(\Phi) \to (\Phi)'$**. Moreover, when the transformation can be built from its infinitesimal limit:**

$$(\Phi)' - (\Phi) = (\delta\Phi)$$

there exist quantities $(\rho, \mathbf{j})$ **constructed from the fields that satisfy continuity equations.**

$$\partial_t \, \rho + \nabla \cdot \mathbf{j} = 0$$

These imply that the "charges" $Q = \int dr(\rho)$ **are time independent constants of motion. Emmy Noether, a protoge of David**

Hilbert, established this connection between (infinitesimal) symmetries and constants of motion. It is a beautiful result and of great utility in unraveling complicated dynamics, since identifying quantities that remain unchanged provides powerful information.

For reasons practical and esthetic, physicists are drawn to field theoretic models that exhibit a large amount of symmetry, and it is even the case that the most successful models possess symmetries that are not seen in nature. Therefore we must resolve the tension between the elegance of great amounts of symmetry and the realism of limited, physically realized symmetries.

To proceed, the unphysical symmetries are gently removed without destroying their attractive features. How to do that? Of course one may simply modify the model so that any unphysical symmetries are explicitly absent. But this would not be a "gentle" solution. A gentler mechanism derives from work on magnetism. Here the equations of motion possess symmetries, but the solutions do not, owing to instabilities of symmetric configurations. This is "spontaneous symmetry breaking". It arises

frequently in condensed matter physics. In particle physics, it had a dramatic, recent and well known realization in the Higgs particle discovery.

An even more subtle effect can eliminate symmetries, without explicit breaking or instabilities. This is known as "anomalous symmetry breaking", or better (since the surprise has worn off) "quantum symmetry breaking." In my first lecture I shall describe this unexpected quantum effect, which removes unwanted symmetries by a topological mechanism, while in two further lectures I discuss other topological effects in quantum field theory.

Quantum Symmetry Breaking

When a dynamical system is examined in order to understand its properties, the rules of analysis must reflect the quantum, rather than the classical, nature of fundamental physics. This is particularly relevant to questions of symmetry. Emmy Noether's analysis of 1918 (summarized above) predates the discovery of quantum physics; she reasoned within rules of classical physics. In retrospect there is no surprise that a change to quantum rules can modify the results.

In summary: A model for physical phenomena may possess a symmetry when its dynamics is analyzed in terms of classical, unquantized, commuting variables. But the symmetry may disappear when the dynamics is quantized and analysis is performed in terms of non-commuting quantum variables. Correspondingly, constants of motion in the unquantized theory are no longer conserved when quantum effects are taken into account.

The example of quantum symmetry breaking that is most widely appreciated arises in a theory of fermions (ψ) interacting with an external potential A_μ. I now explain.

A massless, non-interacting Dirac-Fermi field satisfies the equation

$$i\hbar\,\gamma^\mu \;\frac{\partial}{\partial x^\mu}\;\psi(x) = 0.$$

Summation over a repeated index is implied. ψ is a 4-component column spinor field, x stands for the space-time variables $x^0 = c\,t, x^i = r^i (i = 1, 2, 3)$; the index (μ) ranges over temporal (0) and spatial (i) components; (γ_μ) $(\mu = 0, 1, 2, 3)$ comprise a set of 4×4 Dirac matrices.

For massive fields the equation reads

$$i\hbar\,\gamma^{\mu}\,\frac{\partial}{\partial x^{\mu}}\,\psi(x) - m\,c\,\psi(x) = 0,$$

where m is the mass. (Henceforth we set Planck's constant $\hbar$ **and the velocity of light** c **to unity.)**

The above two equations arise in quantum mechanics; nevertheless we consider them to be "classical" when the field variables are taken to be commuting quantities. Correspondingly, when the field variables are taken to be non-commuting, position-dependent operators, we are dealing with the "quantized" version of the above models.

The two equations, massless and massive, possess a gauge symmetry: If $\psi(x)$ **is a solution, so is** $e^{i\theta}\,\psi(x)$ **where** θ **is an arbitrary constant. And this symmetry is present whether** ψ **is a classical field or a quantum field operator; the corresponding Hamilton's Action is unchanged. As a consequence of this symmetry, the charge**

$$Q \equiv \int d^3\,r\;\psi^{\dagger}\psi$$

is time independent, or equivalently in keeping with Noether's

theorem a charge current 4-vector J^μ

$$J^\mu \equiv \psi^\dagger \gamma^0 \gamma^\mu \psi$$

satisfies a continuity equation

$$\frac{\partial}{\partial x^\mu} J^\mu(x) = 0.$$

[$\psi^\dagger$ **is a 4-component, row spinor, with entries that are complex conjugates of** ψ **(in the unquantized theory) or Hermitian conjugates of** ψ **(in the quantized theory).**]

It is interesting to delve deeper into the matrix structure of these equations. Upon defining the indempotent and Hermitian γ_5 **matrix by**

$$\gamma_5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3, \quad \gamma_5 \gamma_5 = I,$$

we verify that γ_5 **anti-commutes with the Dirac matrices** γ^μ**. It is then easy to check that in the massless case the axial vector current (also called the "chiral current")**

$$J_5^\mu = \psi^\dagger \gamma^0 \gamma^\mu \gamma_5 \psi$$

satisfies a continuity equation as well

$$\frac{\partial}{\partial x^\mu} J_5^\mu(x) = 0$$

and that the axial charge

$$Q_5 \equiv \int d^3r\, \psi^\dagger\, \gamma_5\, \psi$$

is time independent. The additional constant of motion arises as a consequence of Noether's theorem applied to an axial gauge symmetry. The transformation

$$\psi \to \psi' = e^{i\gamma_5\theta}\, \psi = (\mathbf{cos}\,\theta + i\,\gamma_5\,\mathbf{sin}\,\theta)\ \psi,$$

maps solutions into solutions of the massless equation, and this is true whether ψ is a classical field or a quantum field operator.

To encounter anomalies, we enlarge the massless model by introducing a coupling to a vector gauge field A_μ, treated as an externally prescribed quantity, without dynamics. The massless Dirac equation now becomes

$$i\,\gamma^\mu\ \left(\frac{\partial}{\partial x^\mu} + iA_\mu(x)\right)\ \psi(x) = 0.$$

A superficial examination of this system leads to the conclusion that the previous symmetries, continue to hold; indeed the first can be generalized to a "local" gauge symmetry with $\theta(x)$ acquir-

ing a space-time dependence, provided A_μ is also transformed.

$$A_\mu(x) \to A_\mu(x) - \frac{\partial}{\partial x^\mu}\, \theta(x)$$

[When the transformation parameter θ is position independent, as previously considered, the symmetry is a "global" gauge symmetry; otherwise it is "local".]

Correspondingly one would conclude that even in the presence of A_μ that the vector and chiral charges Q and Q_5 remain time-independent and the vector and axial vector currents still satisfy continuity equations.

But these conclusions are valid only if the ψ fields are classical functions and not quantum field operators. For the latter, the problem resides in the fact that the fundamental quantization condition for Dirac-Fermi fields

$$\psi^\dagger_m(t,\mathbf{r})\,\psi_n(t,\mathbf{r}') + \psi_n(t,\mathbf{r}')\,\psi^\dagger_m(t,\mathbf{r}) = \delta_{mn}\,\delta^3\,(\mathbf{r}-\mathbf{r}') \tag{1}$$

implies that the product of $\psi^\dagger$ and ψ at the same space-time point is necessarily singular. [In the above (m, n) label the components

of $\psi^\dagger$ and ψ.] Since the charges and currents involve bilinears of the Dirac-Fermi fields at the same space-time point, they are necessarily ill-defined in the quantum theory. Regularization and renormalization is needed to render the currents well-defined. But it turns out that every regularization/renormalization method in the presence of the vector field A_μ violates the symmetries that are present in the unquantized theory. It is possible to preserve conservation of the vector or axial vector current [or a linear combination of the two] but not both.

Since the preservation of both symmetries is impossible, a choice must be made which one should be preserved. The choice is dictated by the physical context of the theory under examination. Since local gauge symmetries are frequently needed for consistency of the theory (as in the standard model of particle physics) they are the ones whose current conservation must be preserved, while global axial gauge symmetries are abandoned — they become beset by anomalies.

Physical Consequences of Axial Symmetry Anomalies

For the example given above, preserving the local gauge symmetry has the consequence that in the regulated/renormalized quantum field theory the vector charge remains conserved and the vector current continues to satisfy the continuity equation . Correspondingly the axial charge acquires a time dependence and the axial vector current obeys an anomalous continuity equation. Its form is

$$\frac{\partial}{\partial x^\mu}\; J_5^\mu\,(x) = \frac{N}{8\pi^2}\;{}^*F^{\mu\nu}(x)\,F_{\mu\nu}\,(x),$$

where $F_{\mu\nu}$ **is the field strength constructed from** A_μ

$$F_{\mu\nu}(x) \equiv \frac{\partial}{\partial x^\mu}\;A_\nu(x) - \frac{\partial}{\partial x^\nu}\;A_\mu(x)$$

and ${}^*F^{\mu\nu}$ **is its dual.**

$${}^*F^{\mu\nu} \equiv \frac{1}{2}\;\varepsilon^{\mu\nu\alpha\beta}\,F_{\alpha\beta}$$

N **is a numerical constant which is determined by the number and strength of Dirac-Fermi fields coupling to** A_μ**. For the single field of our example,** $N = 1$**. While we have taken** A_μ **to be**

externaly prescribed, it has been shown that the result holds with dynamical A_μ.

The occurrence of the symmetry anomalies leads to a variety of effects in the standard particle physics model.

On the one hand, the standard model appears to possess symmetries that are not present in nature, not even approximately. These classical, global gauge symmetries, if present in the quantized theory, would forbid the decay of a (massless) neutral pion to two photons. But the physical pion's mass can be accurately described as (approximately) vanishing, yet the decay width is not negligible.

$$\Gamma\left(\pi^0 \rightarrow 2\gamma\right) \approx 8.4\, eV$$

Also the same symmetries predict the existence of a neutral pseudo scalar meson, approximately degenerate with the pion. But no such particle has been observed.

It is fortunate that the anomalies in the quantized standard model remove the offending global gauge symmetries. Indeed because the strength of the axial anomaly is known, one can calculate the width for neutral pion decay (for massless pions).

One finds 7.725 eV, or 8.1 eV, when mass corrections are included. This is in excellent agreement with experiment requires that there be three colors of fermions . Thus the axial anomaly in the global gauge symmetry not only determines neutral pion decay and cancels the prediction of an unwanted partner meson, but also gives indirect determination of the number of color degrees of freedom. Furthermore, the standard model possess an anomaly in the continuity equation for the baryon number current, thereby allowing proton decay. While this startling result establishes that in our present theory stability of matter is not absolute, there is no practical significance because the predicted decay rate is negligible

On the other hand, local gauge symmetries must be preserved for consistency of the standard model. This is achieved by adjusting the Fermion content (quarks and leptons) so that possible anomalies cancel. This requirement is met if quarks are matched with leptons, and thus the heaviest "top" quark was predicted to exist once the "bottom" quark was discovered, in order that in the third family of fermion quarks matched the leptons. A simi-

lar anomaly cancellation requirement was found in string theory and led to the revival of that subject.

These physically important effects vividly demonstrate that quantum symmetry anomalies are not obscure pathologies of the quantum mechanical formalism, but describe in a paradoxial-anomalous fashion aspects of natural phenomena.

In addition to elementary particle physics, the anomaly is now recognized for its relevance to condensed matter physics, when a Dirac-type equation exhibits some low energy excitations in materials. This will be explained in greater detail in my subsequent two lectures.

Other instances of anomalous quantum symmetry breaking are clustered around scale symmetry. Presence of this symmetry is signaled by a traceless energy momentum tensor ($\theta_{\mu\nu}$):

$$\textbf{scale symmetry} : (\theta^{\mu}_{\mu}) = 0$$

But in most theories where the trace vanishes classically, quantum effects lead to an anomalous, non-vanishing trace. This happens in curved space-time where (θ^{μ}_{μ}) is given by various geo-

metrical entities. The non-vanishing trace also characterizes the renormalization group, where it involves the beta function.

It is interesting that quantum symmetry breaking acts when mass scales are absent in the classical theory. Is this a hint of further depth to this phenomenon?

Mathematical Connections to Axial Symmetry Anomalies

The discovery of the field theoretic structures associated with axial anomalies seeded an intense interaction between physicists and mathematicians, who for their own purposes had been working with related quantities. Let me explain briefly. (To be accurate, one must pass to Euclidean space and consider non-Abelian quantities, like $SU(2)$.)

First note that $\int d^4x \, {}^*F^{\mu\nu} F_{\mu\nu}$ **is a topological invariant**

$$\Rightarrow \qquad {}^*F^{\mu\nu} F_{\mu\nu}(A) = \partial_\mu C^\mu(A)$$

$$\int d^4x \, {}^*F^{\mu\nu} F_{\mu\nu} = \int d^4x \partial_\mu C^\mu = \int d\Omega_\mu C^\mu$$

$\Rightarrow$ **variation with respect to** A **vanishes**
$\Rightarrow$ **depends only on boundary, quantized values**
$\Rightarrow$ **metric independent**

Next consider the massless Dirac operator in an external vector potential A_μ with eigenvalues λ

$$i\,\gamma^\mu\;(\partial_\mu + A_\mu)\;\psi_\lambda = \lambda\psi_\lambda$$

note that if λ is a eigenvalue belonging to ψ_λ then $-\lambda$ is an eignenvalue of $\gamma_5\psi_\lambda$

zero eigenvalues satisfy $\psi^\pm_{\lambda=0} = \pm\;\gamma_5\psi^\pm_{\lambda=0}; \quad (n\pm)$

$$\begin{aligned} n_+ - n_- \;&\propto\; \int d^4x\,{}^*F^{\mu\nu}F_{\mu\nu}\;\; \textbf{Atiyah-Singer theorem} \\ &\propto\; \int d^4x\langle\partial_\mu J^\mu_5\rangle \end{aligned}$$

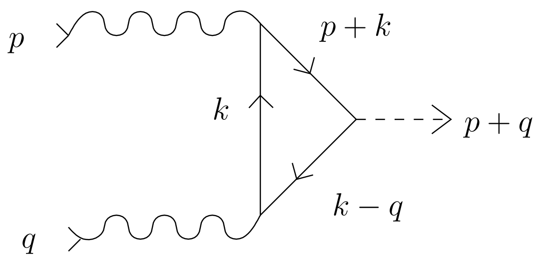
$\langle\partial_\mu J^\mu_5\rangle \propto {}^*F^{\mu\nu}F_{\mu\nu}$ **local version of A-S theorem**

NB ${}^*F^{\mu\nu}F_{\mu\nu} \sim \mathbf{E}\cdot\mathbf{B}$

$$\sim \textbf{Pfaffian } (F_{\mu\nu})$$

Derivation of Chiral Anomaly

$PT \qquad \langle J^\mu(x) J^\nu(y) J_5^\lambda(z) \rangle$



$k \to k + k'$ **Not allowed !!!**

(linear divergence) No unique value

amplitude transverse in 2 of 3 indices

choice up to context

FI **(integral over "classical" fields)**

$\langle \ldots \rangle \propto \int D\mu \, [\exp \, i \, I](\ldots)$

I **classical action** $\to$ **possesses symmetry**

$D\mu$ **not invariant**

choice not obvious

In the late sixties, physicist found a bit of topology making its way into their toolkits. It is truly remarkable that topological effects arise from infinities in the employed formalism, be it at short distances in position space or with large number of quantum excitations in momentum space. The arc of ideas reaches from the two gamma decay of the neutral pion to Witten's most recent "Anomalies Revisited" in Bangalore. I await further ramifications.