Fractional and Majorana Fermions

R. Jackiw (MIT)

A bit of physics history about Dirac and his equation:

Dirac was looking for a relativistic equation for electrons and eventually arrived at the following first-order matrix equation

$$i\,\gamma^{\mu}\,\partial_{\mu}\Psi(x) + m\Psi(x) = 0$$

$$(\boldsymbol{\alpha}\cdot\mathbf{p} + \beta m)\,\Psi = i\,\partial_t\,\Psi$$

$$\boldsymbol{\alpha} = \gamma^0\,\boldsymbol{\gamma},\ \beta = \gamma^0,\ \mathbf{p} = \tfrac{1}{i}\,\boldsymbol{\nabla}$$

Dirac Matrix Equation

$$(\boldsymbol{\alpha}\cdot\mathbf{p}+\beta m)\Psi = i\frac{\partial}{\partial t}\ \Psi$$

$\Psi =$ 4-component complex function (charged excitations)

$\boldsymbol{\alpha},\,\beta \Rightarrow$ four 4×4 “Dirac” numerical matrices

$\mathbf{p}=\frac{1}{i}\boldsymbol{\nabla},\ m=$ mass parameter

Dirac matrices ensure that iteration implies

$$-\frac{\partial^2}{\partial t^2}\ \Psi = -\nabla^2\,\Psi + m^2\,\Psi$$

Dirac equation $=\pm\sqrt{\text{massive wave equation}}$

To expose properties of the Dirac equation, we make the usual decomposition

$$\mathbf{\Psi} = e^{-iEt}\psi \Rightarrow [\boldsymbol{\alpha} \cdot \mathbf{p} + \beta\, m]\psi = E\psi$$

and discover that there exist positive energy solutions $E > 0$, which can describe electrons.

But because "square roots" come in both signs, there exist also negative energy solutions, $E < 0$. These need interpretation because they cannot describe electrons, which carry positive energy. After some hesitation, Dirac concluded that the negative energy solutions correspond to anti-electrons, *viz.* positrons, which soon were discovered.

In a further conceptual leap, Dirac posited that in the ground state all negative energy levels are filled, but nevertheless the charge of the ground state is zero. This means that the Dirac equation is really a many particle equation, where the particles populate the energy levels.

The Dirac equation is a beautiful equation, with a rich hidden physical structure. It goes beyond a single particle interpretation and it predicts new (anti-)particles: the positrons. These characteristics make it a beautiful equation for physics.

Also it is a beautiful equation for mathematics because by using matrices, it succeeds in taking a square root of a second order differential equation.

The mathematical and physical beauty of the Dirac equation suggests to mathematicians and physicists that deformations of the equation may also yield beautiful and interesting results.

Which deformations should we consider?

One alteration that can be made is to reconsider the equation in dimensions different from the three spatial and one time dimension. If we take fewer dimensions we gain the mathematical advantage of simplicity and also have the possibility of describing physical systems that are confined to lower dimensions, for example to a line or to a plane. Such configurations can occur in condensed matter physics. There one encounters situations where the low energy dynamics is well described by a matrix equation, linear in the momenta. Also there may be a constant mass term which separates the positive energy solutions from the negative energy ones by a “gap.” Depending on the nature of the material, the equation may describe excitations on a line, on the plane in addition to those in the three-dimensional bulk.

Dirac Equations
(First-order matrix equations)

$$[\boldsymbol{\alpha} \cdot \mathbf{p} + \beta m]\psi = E\psi$$

continuum solutions $E > |m|$ and $E <$ -$|m|$

"vacuum" :

Particle interpretation: $E < 0$ states filled (antiparticles)
$E > 0$ states empty (particles)

Condensed matter interpretation: $E < 0$ states filled (valence band)
$E > 0$ states empty (conduction band)
m produces gap $2\,|m|$

"vacuum" carries no net charge

One dimension, physics on a line, Dirac matrices realized with 2×2 Pauli matrices

$$1\text{-}d: \quad \alpha = \sigma_2, \quad \beta = \sigma_1, \quad p = \frac{1}{i}\,\frac{d}{dx}$$

A further more profound deformation allows the mass term m to depend on position.

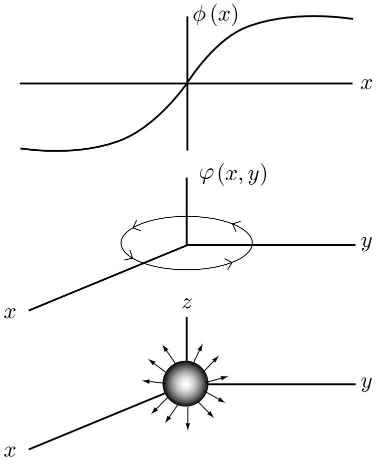
What sort of dependence should we consider?

Surely a weak dependence will produce only insignificant changes from the usual homogenous mass case; we are interested in a significant dependence on position, which could significantly alter the physical situation. Note that the gap $2\,|m|$ depends only on the magnitude of m, and not on its sign. $+m$ produces the same gap as -m. This suggest a deformation of the mass term that interpolates between positive and negative values. In this way we are led to a Dirac equation in the presence of a defect.

(mass term position-dependent, soliton)

$$m \to \varphi(\mathbf{r})$$

$$[\boldsymbol{\alpha} \cdot \mathbf{p} + \beta\, \varphi(\mathbf{r})]\psi = E\psi$$



1-d kink

2-d vortex

3-d magnetic monopole

Find:

continuum solutions $E > 0, E < 0$

AND isolated, normalizable $E = 0$ solution

"mid-gap" state is found by explicit calculation

is guaranteed by index theorems

CENTRAL QUESTION: in "vacuum" is mid-gap state empty or filled, what is its charge?

UNEXPECTED ANSWER: charge $Q = \pm\, \frac{1}{2}$.

Fractional Charge (Analytic Derivation)

Vacuum charge density:

$$\rho(\mathbf{r}) = \int_{-\infty}^{0} dE\, \rho_E(\mathbf{r}) \qquad \rho_E = \psi_E^\dagger \psi_E$$

renormalized charge in soliton background

$$Q = \int d\mathbf{r} \int_{-\infty}^{0-} dE \Big(\rho_E^s(\mathbf{r}) - \rho_E^0(\mathbf{r})\Big)$$

Evaluation simple in the presence of
an energy reflection symmetry:

$\rho_{\text{-}E} = \rho_E$ (charge conjugation)

Fractional Charge Calculation

Completeness: $$\int_{-\infty}^{\infty} dE\, \psi_E^\dagger(\mathbf{r})\psi_E(\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}')$$

$$\Rightarrow \int_{-\infty}^{\infty} dE[\rho_E^s(\mathbf{r}) - \rho_E^0(\mathbf{r})] = 0$$

Conjugation ($\rho_E = \rho_{\text{-}E}$) and zero mode ⇒

$$\int_{-\infty}^{0-} dE\Big(2\rho_E^s(\mathbf{r}) - 2\rho_E^0(\mathbf{r})\Big) + \psi_{E=0}^\dagger(\mathbf{r})\psi_{E=0}(\mathbf{r}) = 0$$

$$\int_{-\infty}^{0-} dE\Big(\rho_E^s(\mathbf{r}) - \rho_E^0(\mathbf{r})\Big) = -\frac{1}{2}\,\psi_{E=0}^\dagger(\mathbf{r})\psi_{E=0}(\mathbf{r})$$

$$Q = -\frac{1}{2}$$

Any dimension!

Empty mid-gap state: $Q = -\frac{1}{2}$

Filled mid-gap state: $Q = +\frac{1}{2}$

Eigenvalue, not expectation value!

Fractional Charge

(Second Quantized Description)

Expansion of quantum Fermi field in presence of defect & zero mode

$$\Psi = \sum_{E>0} (b_E \, \psi_E^s + d_E^\dagger \psi_E^{*s}) + a\psi_{E=0}$$

$$\Psi^\dagger = \sum_{E>0} (b_E^\dagger \, \psi_E^{s*} + d_E \, \psi_E^s) + a^\dagger \, \psi_{E=0}$$

$$a^\dagger a + aa^\dagger = 1$$

How to realize on states:

$a \mid + >=\mid - >, a^\dagger \mid + >= 0,$

$a \mid - >= 0, a^\dagger \mid - >=\mid + >$

$$Q = \frac{1}{2}\int d\,\mathbf{r}(\Psi^\dagger\Psi - \Psi\Psi^\dagger)$$

$$= \frac{1}{2}\sum_{E>0}(b_E^\dagger\, b_E + d_E\, d_E^\dagger - b_E\, b_E^\dagger - d_E^\dagger\, d_E) + \frac{1}{2}(a^\dagger\, a - a\, a^\dagger)$$

$$= \sum_{E>0}(b_E^\dagger\, b_E - d_E^\dagger\, d_E) + a^\dagger\, a - \frac{1}{2}$$

$$Q\,|- > = -\frac{1}{2}\,|- >,$$

$$Q\,|+ > = +\frac{1}{2}\,|+ >$$

eigenvalue !

One Dimensional Example (Polyacetylene)

Dirac equation with varying mass $[\sigma^2 p + \sigma^1 \varphi(x)]\psi = E\psi$

$$E \lessgtr 0 \quad \left[\begin{pmatrix} 0 & -\frac{d}{dx} \\ \frac{d}{dx} & 0 \end{pmatrix} + \begin{pmatrix} 0 & \varphi(x) \\ \varphi(x) & 0 \end{pmatrix}\right] \begin{bmatrix} \psi^u_E \\ \psi^l_E \end{bmatrix} = E \begin{bmatrix} \psi^u_E \\ \psi^l_E \end{bmatrix}$$

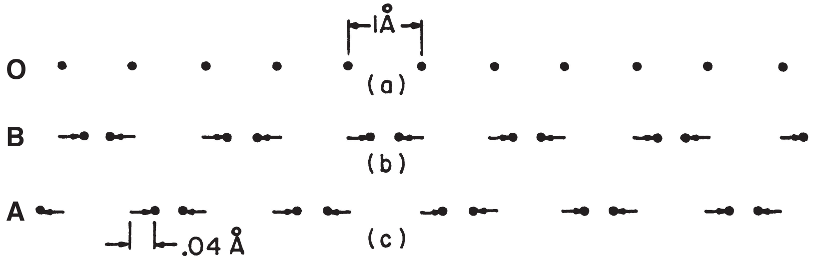
$$E = 0 \qquad \begin{pmatrix} 0 & -\frac{d}{dx} + \varphi(x) \\ \frac{d}{dx} + \varphi(x) & 0 \end{pmatrix} \begin{bmatrix} \psi^u_0 \\ \psi^l_0 \end{bmatrix} = 0$$

$$\psi_0^{u,l} = N \exp \mp \int^x dx' \, \varphi(x')$$

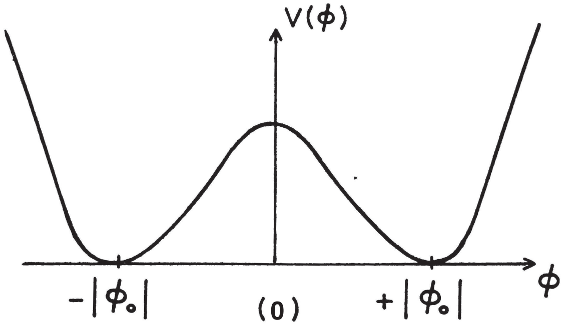
Normalizable provided $\varphi(x)$ has kink profile.

Rebbi & RJ, *PRD* **13**, 3398 (76)
Su, Schrieffer & Heeger, *PRL* **42**, 1698 (79)
[high conductivity in polymers]

Peierls' Instability in Polyacetylene

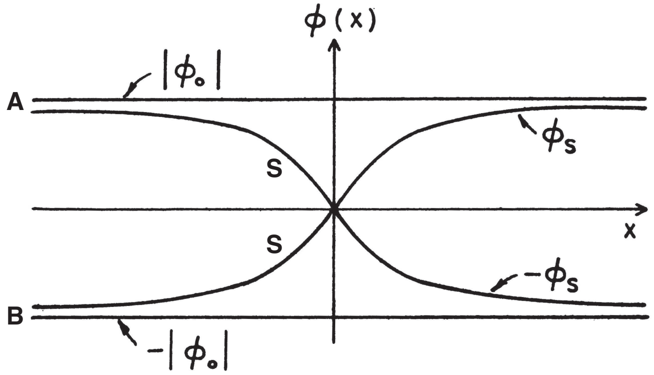


Energetics of Polyacetylene Phonon Field



Energy density $V(\phi)$, as a function of a constant phonon field ϕ. The symmetric stationary point, $\phi = 0$, is unstable. Stable vacua are at $\phi = +|\phi_0|$, (A) and $\phi = -|\phi_0|$, (B).

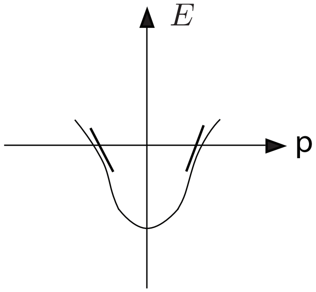
Profiles of Phonon Field



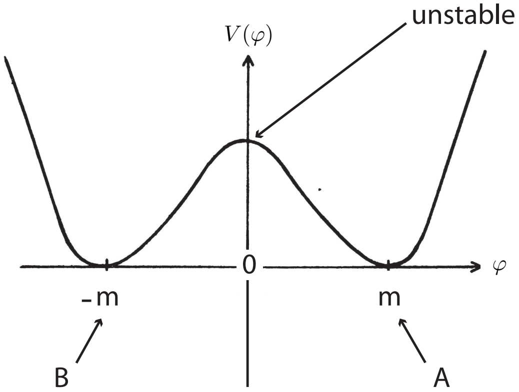
The two constant fields, $\pm \mid \phi_0 \mid$, correspond to the two vacua (A and B). The kink-soliton fields, $\pm\phi_s$, interpolate between the vacua and represent domain walls.

Polyacetylene realization of $(1-d)$ "Dirac" equation

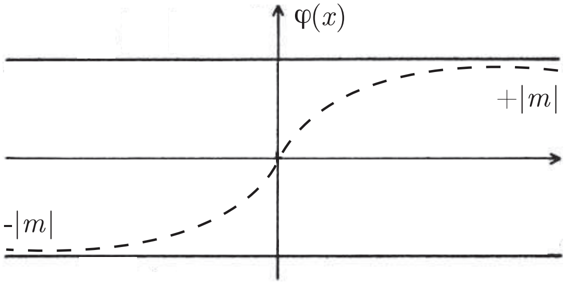
Kinetic term: linearization at Fermi level



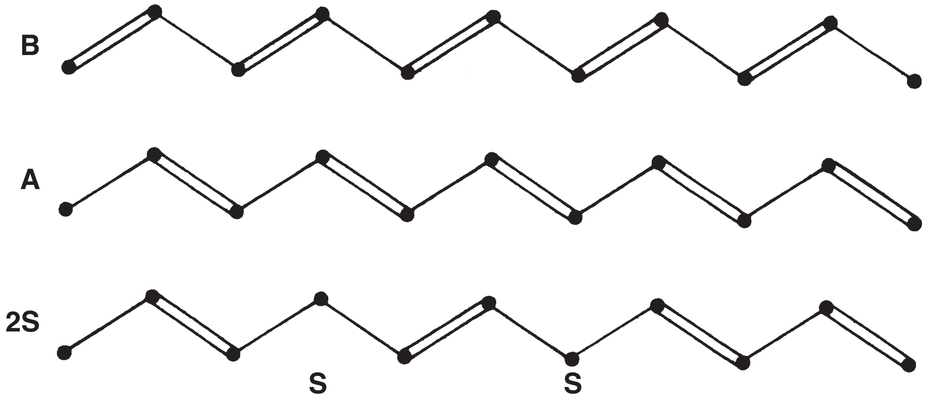
Potential term:
Peierls' instability
energy profile seen
by phonon field



configurations
of phonon
field φ



Fractional Charge Schematic



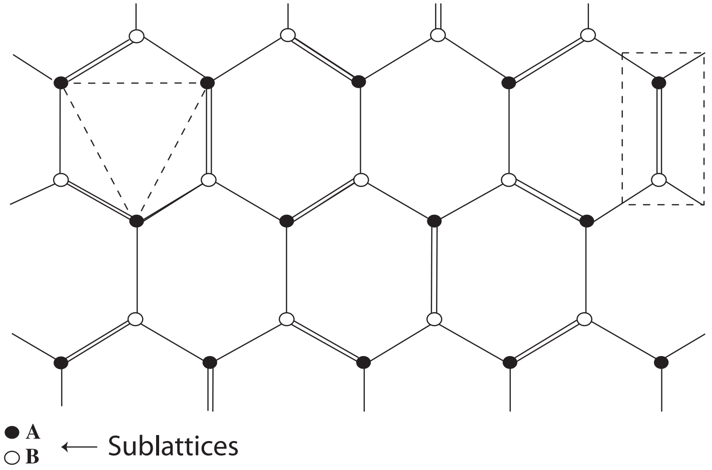
Two soliton state carries one fewer link relative to no-soliton vacuum A.

Separate solitons to $\infty \Rightarrow$
split quantum numbers of link $\Rightarrow$
fermion number fractionalization!

$(2 - d) \Rightarrow$ Fractional charge in quantum Hall effect inspired by solitons result, but different mechanism.

Graphene Realization of (2-d) Dirac Equation

Graphene hexagonal lattice



Kinetic term:

Linearization at Fermi level
2 “Dirac points” per sublattice (conduction and valence bands meet)

⇒ 4 × 4 Dirac Hamiltonian in 2 spatial dimensions

Wallace, *PR* **71**, 662 (47); Semenoff, *PRL* **53**, 2449 (84);
Gaim & Novoselov, Nobel Prize (10)

Potential term: ? Speculate: “Kekule’ distortion”

Dirac Hamiltonian

$$H = \underbrace{\psi^\dagger \boldsymbol{\alpha} \cdot \mathbf{p}\, \psi}_{} + \underbrace{\psi^\dagger \beta \left[\varphi_{re} - i\, \varphi_{im}\, \gamma_5\right]}_{} \psi \qquad (4 \times 4)$$

$$\varphi \equiv \varphi_{re} + i\, \varphi_{im}$$

$$\boldsymbol{\alpha} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix},\ \beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},\ \gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},\ \mathbf{p} = \frac{1}{i}\boldsymbol{\nabla}$$

(All vectors are 2-dimensional)

linearization
at Fermi level

Kekulé distortion
φ constant: $\varphi_0 \Rightarrow$ mass gap
φ soliton/vortex: $\varphi_s = |\varphi(r)| e^{i\theta}, \varphi(0) = 0, |\varphi(\infty)| = \varphi_0$
$\Rightarrow$ mid-gap state

Conclusion : $Q = \pm\frac{1}{2}$ (existence proof!)

Hou, Chamon & Mudry, PRL **98**, 186809 (07) [cond-mat/0609740]

Majorana Equation

electrically charged particles:

particle is different from anti-particle

created by complex field

electrically neutral particles:

particle can be identified with its anti-particle

created by real field

$$\left.\begin{array}{lll}\text{e.g.} & \text{neutral pion} & (S=0)\\ & \text{photon} & (S=1)\\ & \text{graviton} & (S=2)\end{array}\right\rangle \text{all bosons}$$

Majorana fermion = neutral fermion

Majorana Matrix Equation

$$(\boldsymbol{\alpha}\cdot\mathbf{p}+\beta m)\Psi = i\,\frac{\partial}{\partial t}\,\Psi$$

Ψ real (neutral excitations)

$\mathbf{p}=\frac{1}{i}\nabla$ imaginary

$$\left.\begin{array}{l}\boldsymbol{\alpha}\ \text{real} = \boldsymbol{\alpha}^*\\ \beta\ \text{imaginary} = -\boldsymbol{\beta}^*\end{array}\right\}\ \text{Majorana Representation}$$

Majorana Representation

$$\alpha^1_M = \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix} \; \alpha^2_M = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \; \alpha^3_M = \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}$$

$$\beta_M = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix} \qquad \Psi^*_M = \Psi_M$$

Majorana in arbitrary representation

$$C\boldsymbol{\alpha}^* C^{-1} = \boldsymbol{\alpha},\; C\beta^* C^{-1} = -\beta \quad C\Psi^* = \Psi$$

e.g. Weyl $\boldsymbol{\alpha} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix} \; \beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \; C = \begin{pmatrix} 0 & -i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix}$

$$\begin{pmatrix} \boldsymbol{\sigma}\cdot\boldsymbol{p} & m \\ m & -\boldsymbol{\sigma}\cdot\boldsymbol{p} \end{pmatrix} \begin{pmatrix} \psi \\ \chi \end{pmatrix} = i\frac{\partial}{\partial t} \begin{pmatrix} \psi \\ \chi \end{pmatrix}$$

$$C\Psi^* = \Psi \quad \Rightarrow \quad \chi = i\sigma^2 \psi^*$$

$$\boldsymbol{\sigma}\cdot\boldsymbol{p}\,\psi + i\sigma^2 m\,\psi^* = i\frac{\partial}{\partial t}\,\psi \qquad (2\times 2)$$

ψ mixes with ψ^*

NB $C = I$ in Majorana representation

Majorana Equation (2 component)

$$\boldsymbol{\sigma}\cdot\mathbf{p}\,\psi + i\sigma^2\,m\,\psi^* \;= i\frac{\partial}{\partial t}\,\psi$$

NB. Dirac mass term: preserves quantum numbers (charge, particle number)

Majorana mass term: does not preserve any quantum numbers

$\Rightarrow$ no distinction between particle and anti-particle since there are no conserved quantities to tell them apart, particle is its own anti-particle

Dirac field operator

$$\Psi = \sum_{E>0}\left(a_E\,e^{-iEt}\,\Psi_E + b_E^\dagger\,e^{iEt}\,C\,\Psi_E^*\right)$$

Majorana field operator

$$\Psi = \sum_{E>0}\left(a_E\,e^{-iEt}\,\Psi_E + a_E^\dagger\,e^{iEt}\,C\,\Psi_E^*\right)$$

anti-particle operators $(b, b^\dagger)$ have disappeared

Are there Majorana fermions in Nature?

neutrinos?

– recent development in neutrino physics

experimental observation of neutrino oscillations ⇒

- neutrinos have mass (< 0.1eV)
- lepton number is not conserved separately for each flavor.

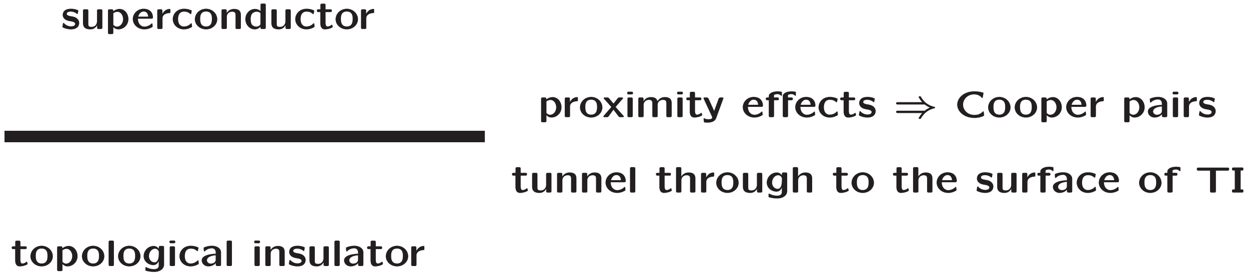
⇒ they could be Majorana fermions

Hypothetical Majorana fermions:

- supersymmetry – supersymmetric partners of photon, neutral Higgs boson, etc. are necessarily Majorana fermions

- cosmology – dark matter candidates

Majorana fermions in superconductor in contact with a topological insulator



superconductor

proximity effects ⇒ Cooper pairs

tunnel through to the surface of TI

topological insulator

Hamiltonian density for the model:

$$H = \psi^* \; (\boldsymbol{\sigma} \cdot \frac{1}{i} \; \boldsymbol{\nabla} - \mu) \; \psi + \frac{1}{2} \; (\triangle \psi^* \, i \, \sigma_2 \, \psi^* + h.c.)$$

$$\psi = \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix}, \boldsymbol{\sigma} = (\sigma_1, \sigma_2),$$

μ is chemical potential, $\triangle$ is the order parameter

$\triangle$ may be constant: $\triangle = \triangle_{0j}$

or take vortex profile: $\triangle(\mathbf{r}) = v(r) e^{i\theta}, v(0) = 0, v(\infty) = \triangle_0$.

Equation of motion: $i \, \partial_t \, \psi = (\boldsymbol{\sigma} \cdot \mathbf{p} - \mu) \; \psi + \triangle \, i \, \sigma^2 \, \psi^*$

In the absence of μ, and with constant $\triangle$, the above system is a (2+1)-dimensional version of the (3+1)-dimensional, two component Majorana equation!

⇒ governs chargeless spin $\frac{1}{2}$ fermions with Majorana mass $|\triangle|$.

Zero Mode

In the presence of a single vortex order parameter $\Delta(\mathbf{r}) = v(r)e^{i\theta}$ there exists a zero-energy (static) isolated mode
(Fu & Kane, *PRL* **100**, 096407 (08); Rossi & RJ *NPB* **190**, 681 (81))

$$\psi_0 = N \begin{pmatrix} J_0(\mu r)\,\mathrm{exp}\,\{-i\pi/4 - V(r)\} \\ J_1(\mu r)\,\mathrm{exp}\,\{i(\theta + \pi/4) - V(r)\} \end{pmatrix}$$

N real constant, $V'(r) = v(r)$

Majorana field expansion:

$$\Psi = \underbrace{\ldots\ldots\ldots\ldots\ldots}_{E\neq 0 \text{ modes}} + a\,\Psi_0$$

where zero mode operator a satisfies

$$\{a, a^\dagger\} = 1, a^\dagger = a \Rightarrow a^2 = 1/2$$

[Chamon, Nishida, Pi, Santos & RJ; *PRB* **81**, 224515 (10)]

(i) Two 1-dimensional realizations: take vacuum state to be eigenstate of a, with possible eigenvalue $\pm 1/\sqrt{2}$.

$$a\,|0\pm\rangle = \pm\,\frac{1}{\sqrt{2}}\;|0\pm\rangle$$

There are two ground states $|0+\rangle$ and $|0-\rangle$. Two towers of states are constructed by repeated application of $a_E^\dagger$. No operator connects the two towers.

Fermion parity is broken because a is a fermionic operator. Like in spontaneous breaking, a vacuum $|0+\rangle$ or $|0-\rangle$ must be chosen, and no tunneling connects to the other ground state.

(ii) One 2-dimensional realization: vacuum doubly degenerate $|1\rangle\,,|2\rangle$, and a connects the two vacua.

$$a\,|1\rangle = \tfrac{1}{\sqrt{2}}\;|2\rangle$$

$$a\,|2\rangle = \tfrac{1}{\sqrt{2}}\;|1\rangle$$

Two towers of states are constructed by repeated application of $a_E^\dagger$. a connects the towers. Fermion parity is preserved.

We shall assume that fermion parity is preserved, and adopt second possibility

- Curious fact in (1-d)

  total $\mathcal{L}$ for scalar kink $\oplus$ fermions

  $\mathcal{L} = \frac{1}{2}\,\partial_\mu \Phi\, \partial^\mu \Phi + \frac{\mu}{2}^2 \Phi^2 - \frac{\lambda}{8}^2 \Phi^4 \quad + i \bar{\Psi}\, \gamma^\mu\, \partial_\mu\, \Psi - g\, \Phi\, \bar{\Psi}\Psi$

  $\mathcal{L}$ possesses SUSY for $g = \lambda, \Psi$ Majorana

  Center anomaly in SUSY algebra ⇒ fermion parity can be absent.

[Losev, Shifman & Vainshtein, *PLB* **522**, 327 (01)

Any relevance for condensed matter?

Semenoff & Sodano, *EJTP* **10**, 57 (08)]

Multiple Vortices

With N vortices, governed by operators $a_1, a_2, \ldots, a_N$ that satisfy

$$\{a_i, a_j\} = 2\,\delta_{ij} \qquad \text{(Clifford algebra)}$$

one can show that one needs

$$\mathcal{N} = 2^{\frac{N}{2}} \quad \text{states for even } N$$

$$\text{and} \quad \mathcal{N} = 2^{\frac{N+1}{2}} \quad \text{states for odd } N$$

$N = 1$	$\mathcal{N} = 2$	σ_1 or σ_2 (not σ_3)	(2×2)
$N = 2$	$\mathcal{N} = 2$	σ_1 and σ_2 (not σ_3)	(2×2)
$N = 3$	$\mathcal{N} = 4$	α_1 α_2 α_3 or β (not diagonal)	(4×4)
$N = 4$	$\mathcal{N} = 4$	α_1 α_2 α_3 and β (not diagonal)	(4×4)
	etc.		

Clifford algebra, with a restriction: use for a_i Pauli, Dirac, . . . , matrices excluding diagonal one since it would correspond to diagonalizing a mode operator and would produce fermion parity violation [Pi & RJ, *PRB* **85**, 033102 (12)]