BF theories and graphene

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Outline

1. TQFT & braid statistics

2. From TQFT to Turaev–Viro state sum models
   - Double CS–BF with a $\Lambda$-term
   - Double CS–TV

3. BF effective action in graphene
3D Topological Quantum Field Theories (TQFTs)

- are gauge theories characterized by a classical action that does not depend on the ‘metric’ of the underlying 3D spacetime manifold
- Here the gauge group $G$ is non–Abelian, typically $SU(2)$
- The gauge field $A^a_\mu$ (the ‘vector potential’) is a connection 1-form ($\mu = 0, 1, 2$) with value in the Lie algebra $G$ of $G$ ($a = 1, 2, \ldots$, rank ($G$))
- The field strength $F_{\mu\nu}$ is the curvature 2-form

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \to F := dA + A \wedge A \]

Types: Chern–Simons (CS)–Witten and BF
- related to 3D and (2+1)D gravity, Witten (1989);
- suitable to describe the IR behavior of some 2D electron systems;
- support ‘Topological Quantum Computation’
Physical restrictions: PT–invariant models in (2+1)D or 3D with

- a mass gap
- topologically protected degenerate ground states
- non–chiral quasi–particle excitations, called (non-Abelian) ‘anyons’, characterized by a fractionary or ‘braid’–like statistics
Anyon dynamics bears on the existence of unitary representations of the 'braid group' included or conjectured in

- **Theoretical background models (*)&nbsp;**
  - 3D Quantum Topological field theories (Chern–Simons, BF)
  - 2D Boundary Conformal Field Theories (WZW)
  - 2D Lattice gauge theories

- **Possible experimental settings**
  - 2D electron systems in B field (Fractional Q Hall Effect)
  - Cold atoms in optical lattices
  - Bose–Einstein condensates
  - Topological insulators
  - Graphene

(*) [Das Sarma et al *Non Abelian Anyons and TQC*, arXiv:0707.1889]
The physical restrictions select in particular

♦ Double quantum Chern–Simons 3D environment ($SU(2)_k$, $k \geq 3$), and associated *ad hoc* discretized models on 2D ‘boundaries’ [Freedman et al, *A class of PT-invariant topological phases of interacting electrons*, arXiv:cond-mat/0307511]

However, as will be shown in the following

♦ Double CS ↔ BF with cosmological constant or $\Lambda$ term

♦ BF, $\Lambda$ ↔ $SU(2)_q$ (Turaev–Viro) state sums at $q = \exp(2\pi i/k)$ for 3D triangulated ‘colored’ manifolds induces *naturally* 2D ‘dual’ lattice models with the required features [Z Kádár, A M, M Rasetti, *Microscopic description of 2D topological phases, duality, and 3D state sum models*, arXiv:0806.3883; 0907.3724]
Quantization of (double) Chern–Simons 3D field theory

- proceeds through the (Euclidean) path integral formalism; for closed 3-manifolds the quantum generating functional is a topological invariant of the manifold

- Gauge invariant (and Diff-invariant) quantum observables are vacuum expectation values (v.e.v.) of Wilson operators defined for oriented knots presented as (closed) geometric braids decorated with irreps of $SU(2)_k$

- Up to suitable (2+1) decompositions, the associated unitary representations of the braid group provide a consistent scheme to address ‘anyons dynamics’
An $SU(2)$–colored oriented braid configuration $\mathcal{B}$ (embedded e.g. in $\mathbb{R}^3$): a WZW theory is induced on the 2D boundaries with primary fields associated to Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ with real dimensions $N = (2j + 1) (2k + 1) (2l + 1)$. 

$\{j, k, l = 0, 1/2, ..., (k - 1)/2\}$ label irreps of $SU(2)_k$, ($\rightarrow$ next lectures)
Classical actions ($G = SU(2)$); closed oriented 3–manifold $M^3$;
$A$: connection; $F = dA + A \wedge A$; $B$: B–field; $\mu, \cdots = 0, 1, 2$;
trace over Lie algebra indices

$$S_{BF, \Lambda} = \int d^3x \epsilon^{\lambda\mu\nu} \text{Tr} \left( B_\lambda F_{\mu\nu} + \frac{\Lambda^2}{3} B_\lambda B_\mu B_\nu \right)$$

$$S_{BF, \Lambda} = [S_{CS}(A^+) - S_{CS}(A^-)]$$

$$S_{CS}(A^{\pm}) = \frac{k}{4\pi} \int d^3x \epsilon^{\lambda\mu\nu} \text{Tr} \left( A^{\pm}_\lambda \partial_\mu A^{\pm}_\nu + \frac{2}{3} A^{\pm}_\lambda A^{\pm}_\mu A^{\pm}_\nu \right)$$

- $\Lambda^2 = (4\pi/k)^2$, called ‘cosmological constant’, is in front of a contribution to the action which, in the gravitational context, is given by the volume of the manifold under $B_\mu \to$ the dreibein;
- $k$ (CS coupling constant) constrained to be an integer by the quantization procedure [Deser, Jackiw, Templeton, PRL & Ann. Phys. 1982]
The BF connection and the B-field are

\[ A_\mu = \frac{1}{2} (A_\mu^+ + A_\mu^-) \]

\[ B_\mu = \frac{k}{8\pi} (A_\mu^+ - A_\mu^-) \]

The generating quantum functional is formally given by the path integral

\[ Z_{BF,\Lambda}[M^3] = \int \mathcal{D}A\mathcal{D}B \exp \{ i S_{BF,\Lambda}(A, B) \} \]

- There could be another coupling constant in front of the action, but it is just a numerical factor owing to the relation between \( \Lambda \) and \( k \)
Gauge–invariant (and Diff–invariant) quantum observables associated with oriented closed **knotted** curves $C^1$ embedded in $M^3 \leftrightarrow$ v.e.v. of **Wilson loop** operators

$$Z_{BF, \Lambda}[M^3, C] = \int \mathcal{D}A \mathcal{D}B \exp \{ i S_{BF, \Lambda}(A, B) \} \text{Tr Hol}(A \pm \Lambda B)$$

normalized by $(Z_{BF, \Lambda}[M^3])^{-1}$, where the holonomies

$$\text{Hol}(A \pm \Lambda B) := \exp \left\{ i \oint_C (A_\mu \pm \Lambda B_\mu) \, dz^\mu \right\}$$

are evaluated (up to path ordering) along the curve $C$ parametrized by local coordinates $z^\mu$

1knots and links, namely collections of disjoint interlaced knots
Not difficult to prove that at the formal level

\[ Z_{BF, \Lambda}[M^3] = Z_{CS, k}[M^3] \overline{Z_{CS, k}[M^3]} \]

where \( Z_{CS, k}[M^3] \) is the CS–Witten generating functional (a topological invariant) for the closed manifold \( M^3 \) and the complex conjugate \( Z \) refers to \( M^3 \) with the opposite orientation.

Observables:

\[ Z_{BF, \Lambda}[M^3, C] =? Z_{CS, k}[M^3, C] \overline{Z_{CS, k}[M^3, C]} \]

where \( Z_{CS, k}[M^3, C] \) represents a polynomial invariants of knots (links) of the Jones’ type [V Jones 1985; E Witten 1989]
Double CS–BF with a $\Lambda$-term

\[ Z_{\text{BF}, \Lambda}[M^3] \leftrightarrow Z_{\text{CS}, k}[M^3] \] \[ \overline{Z_{\text{CS}, k}[\overline{M^3}]} \] again

- The proof of interest here relies on an intermediate correspondence with the TV state sum

Preliminary remark: any closed smooth 3D manifold $M^3$ can be presented as the complement of a framed knot $K$ (surgery knot) embedded in the 3-sphere $S^3$. The resulting 'quantum invariant' is a topological invariant [Reshetikhin–Turaev, 1991]) expressed as a weighted sum of colored Jones polynomials of the knot $K$ and $Z_{\text{CS}}$ (Witten 1989) = $Z_{RT}$
Double CS–BF with a $\Lambda$-term

\[ Z_{BF, \Lambda}[M^3, C] = Z_{CS, k}[M^3, C] \overline{Z_{CS, k}[M^3, C]} \]

- Knot invariants arising in the CS environment are essentially the same as those found in double CS and thus in BF, $\Lambda$ [A Cattaneo et al, J. Math. Phys. 36 (1995) 6137]
- Field-theoretical background: E Guadagnini
  *The link invariants of the Chern–Simons field theory*, 1993


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Any closed Piecewise–Linear (PL) manifold $M^3$ can be presented as a triangulation $T^3(q; j_1, j_2, \ldots, j_N)$ made of colored tetrahedra glued along their triangular faces (Here colors are labels of irreps of $SU(2)_q$ at $q$ a root of unity, $q = \exp(2\pi i/k)$)

The associated partition function or state sum is a topological (PL) invariant [Turaev–Viro, 1992], the regularized counterpart of Ponzano–Regge model for Euclidean 3D gravity
The Turaev–Viro state sum for a closed 3-manifold

- **Vertex** → weight $w(q)^{-2}$
- **Edge** $J$ →
  - $q$–dimension $[2J + 1]_q$
- **Tetrahedron** →
  - $q - 6j$-symbol
    \[ \left\{ \begin{array}{ccc} J_1 & J_2 & J_3 \\ J_4 & J_5 & J_6 \end{array} \right\}_q \]

\[
Z_{TV}[M^3, q] = \sum_{(T^3(J) \text{ of } M^3)} \prod_{\text{vert}} w(q)^{-2N_0} \prod_{\text{edges}} [2J + 1]_q \prod_{\text{tetra}} \{6j\}_q
\]

$N_0$: number of vertices; sum over all colored triangulations
For each fixed value of $q = \exp(2\pi i/k)$
• Proof based on the *skein calculus* starting from the Reshetikhin–Turaev invariant quoted above [K Walker 1991, J Roberts, 1995]: from $M^3 \cup \overline{M}^3$ presented as the complement of a framed, colored knot, a colored triangulation is generated. *Invariance* relies on the improvement of suitable combinatorial moves, translated into algebraic identities for the $q$-$6j$ symbols (pentagon and hexagon relations in the language of braided tensor categories) & 'Kirby' moves (…)

• Analogues of colored knot invariants (observables in quantum CS) can be defined, and the resulting *combinatorial expressions* agree with the path integral formulas, up to normalization
On the basis of the mutual correspondence with the double quantum CS (in the oriented case) the conclusion is:

\[ Z_{BF, \Lambda}[M^3] = Z_{TV}[M^3, q] ; \quad \Lambda = 4\pi/k \]

- Actually the result has been proven directly for the 3D Euclidean (gravitational + cosmological term) action in the first order formalism (with \( A \) the spin connection and \( B \) the dreibein)
- As discussed above the same correspondence holds for observables, in particular for \( SU(2) \)-colored knot invariants, for each value of \( q \)
- Extensions to colored triangulated 3-manifolds with boundary are crucial e.g. to characterize effective actions of topological nature in (2+1) lattice systems
At room temperature, near the Fermi points, electrons exhibit a relativistic behavior → (2+1)D massless Dirac equation

\[ H = -i\hbar v_f \partial \nabla \]

\(v_f\) is the Fermi velocity, \(-10^6\) m/s

\[ i \gamma^\mu \partial_\mu \psi = 0 \]

(Chiral representation for the gamma matrices)

Bi–spinor (A,B refer to the two triangular sublattices):

\[
\psi = \begin{pmatrix}
\psi_A^+ \\
\psi_B^+ \\
\psi_A^- \\
\psi_B^-
\end{pmatrix}
\]

- One way for have a mass gap in this material is to add a chemical potential \( \mu \)
- Topological defects in the honeycomb lattice generate vortices \( \rightarrow \) coupling with \( U(1) \times U(1) \) gauge fields \( a_\mu \) and \( b_\mu \) identified as the ordinary em field and the chiral gauge field [Ryu et al, 2009]
The low-energy action $S(a, b, \psi\bar{\psi})$ for coupled fermions in graphene is

$$S = -\beta \int d^3x \bar{\psi}^f (i\gamma^\mu \partial_\mu - \gamma^\mu a_\mu - \gamma^5 \gamma^\mu b_\mu - \mu \gamma_0) \psi_f$$

where $\beta$ is a constant and the index $f = 1, 2$ takes into account the (real) spin degeneracy of particles.

**Further steps**

- The replica method takes care of disorder–averaged physical quantities
  $\Rightarrow$ the gauge fields are 1-forms in (the Lie algebra of) $U(N) \times U(N)$
- Wick rotation on the background manifold
- Fermionic degrees of freedom integrated out
- Regularization (Pauli-Villars)
- Level-rank duality in gauge theories: $N \leftrightarrow k$ (CS coupling constant)
For $\beta$ a positive integer the resulting action is a $U(2)\beta \times U(2)\beta$ Chern–Simons action, equivalent to the BF action

$$S_{BF,\Lambda} = \int d^3x \epsilon^{\lambda\mu\nu} \text{Tr} \left( B_\lambda F_{\mu\nu} + \frac{\Lambda^2}{3} B_\lambda B_\mu B_\nu \right)$$

(1)

where

$$A_\mu = a_\mu ; \quad B_\mu = \frac{\beta}{4\pi} b_\mu$$

- In bold: non–Abelian fields
- Natural 1–1 mapping with the gauge fields of the fermionic action, $a_\mu \leftrightarrow$ connection and shifted $b_\mu \leftrightarrow$ the chiral field responsible for vortices
- Gauge invariant quantum observables $\leftrightarrow$ anyonic–type excitations (?)
Outlook (BF-TV environment)

- Beyond the monolayer flat graphene: effective topological actions for curved and multilayer graphene
- Beyond zero-temperature: perturbative expansions of quantum observables (?)
- **2D Boundaries & 1D embedded structures**: improving the comprehension of interconnections (BF with \( \Lambda \) term \( \leftrightarrow \) TV) also in view of applications

\( \Rightarrow \) Learn more on TV state sum models (next lectures), thus circumventing the path integral formalism of standard TQFT