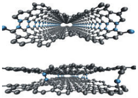
BF theories and graphene

Annalisa Marzuoli

Dipartimento di Matematica, Pavia University

ISS27 GRAPHENE



Prague, September 14-18, 2015

Outline

1. TQFT & braid statistics
2. From TQFT to Turaev–Viro state sum models
   - Double CS–BF with a Λ-term
   - Double CS–TV
3. BF effective action in graphene

3D Topological Quantum Field Theories (TQFTs)

• are gauge theories characterized by a classical action that does not depend on the 'metric' of the underlying 3D spacetime manifold
• Here the gauge group G is non–Abelian, typically $SU(2)$
• The gauge field A_μ^a (the 'vector potential') is a connection 1-form ($\mu = 0, 1, 2$) with value in the Lie algebra $\mathcal{G}$ of G
($a = 1, 2, \ldots,$ rank $(\mathcal{G})$)
• The field strength $F_{\mu\nu}$ is the curvature 2-form

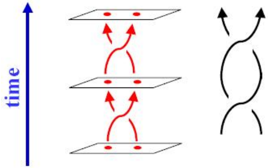
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \to F := dA + A \wedge A$$

Types: Chern–Simons (CS)–Witten and BF

□ related to 3D and (2+1)D gravity, *Witten (1989)*;
■ suitable to describe the IR behavior of some 2D electron systems;
□ support 'Topological Quantum Computation'

Physical restrictions: PT–invariant models in (2+1)D or 3D with

- a mass gap
- topologically protected degenerate ground states
- non–chiral quasi–particle excitations, called (non-Abelian) 'anyons', characterized by a fractionary or 'braid'–like statistics



Anyon dynamics bears on the existence of unitary representations of the 'braid group' included or conjectured in

- **Theoretical background models** (*)
  - 3D Quantum Topological field theories (Chern–Simons, BF)
  - 2D Boundary Conformal Field Theories (WZW)
  - 2D Lattice gauge theories
- **Possible experimental settings**
  - 2D electron systems in B field (Fractional Q Hall Effect)
  - Cold atoms in optical lattices
  - Bose–Einstein condensates
  - Topological insulators
  - Graphene

(*) [Das Sarma et al *Non Abelian Anyons and TQC*, arXiv:0707.1889]

The physical restrictions select in particular

♦ *Double* quantum Chern–Simons 3D environment ($SU(2)_k$, $k \geq 3$), and associated *ad hoc* discretized models on 2D 'boundaries' [Freedman et al, *A class of PT-invariant topological phases of interacting electrons*, arXiv:cond-mat/0307511]

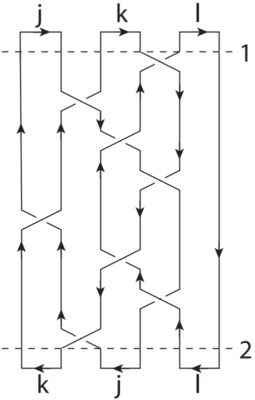
However, as will be shown in the following

♦ Double CS ↔ BF with cosmological constant or Λ term

♦ BF, $_\Lambda$ ↔ $SU(2)_q$ (Turaev–Viro) state sums at $q = \exp(2\pi i/k)$ for 3D triangulated 'colored' manifolds induces *naturally* 2D 'dual' lattice models with the required features [Z Kádár, A M, M Rasetti, *Microscopic description of 2D topological phases, duality, and 3D state sum models*, arXiv:0806.3883; 0907.3724]

Quantization of (double) Chern–Simons 3D field theory

- proceeds through the (Euclidean) path integral formalism; for closed 3-manifolds the quantum generating functional is a topological invariant of the manifold
- Gauge invariant (and Diff-invariant) quantum observables are vacuum expectation values (v.e.v.) of **Wilson operators** defined for oriented knots presented as (closed) geometric braids decorated with *irreps* of $SU(2)_k$
- Up to suitable (2+1) decompositions, the associated unitary representations of the braid group provide a consistent scheme to address ′anyons dynamics′



An $SU(2)$–colored oriented braid configuration $\mathcal{B}$ (embedded *e.g.* in $\mathbb{R}^3$):
a WZW theory is induced on the 2D boundaries with primary fields associated to Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ with real dimensions
$N = (2j+1)\ (2k+1)\ (2l+1)$.
$\{j, k, l = 0, 1/2, ..., (k-1)/2\}$ label irreps of $SU(2)_k$, ($\rightarrow$ next lectures)

Classical actions ($G = SU(2)$); closed oriented 3–manifold M^3); A: connection; $F = dA + A \wedge A$; B: B–field; $\mu, \cdots = 0, 1, 2$; trace over Lie algebra indices

$$S_{BF,\Lambda} = \int d^3x \, \epsilon^{\lambda\mu\nu} \, \mathrm{Tr} \left(B_\lambda F_{\mu\nu} + \frac{\Lambda^2}{3} B_\lambda B_\mu B_\nu \right)$$

$$S_{BF,\Lambda} = \left[S_{CS}(A^+) - S_{CS}(A^-) \right]$$

$$S_{CS}(A^\pm) = \frac{k}{4\pi} \int d^3x \, \epsilon^{\lambda\mu\nu} \, \mathrm{Tr} \left(A^\pm_\lambda \partial_\mu A^\pm_\nu + \frac{2}{3} A^\pm_\lambda A^\pm_\mu A^\pm_\nu \right)$$

• $\Lambda^2 = (4\pi/k)^2$, called 'cosmological constant', is in front of a contribution to the action which, in the gravitational context, is given by the volume of the manifold under $B_\mu \to$ the dreibein;
• k (CS coupling constant) constrained to be an integer by the quantization procedure [*Deser, Jackiw, Templeton, PRL & Ann. Phys. 1982*]

The BF connection and the B-field are

$$A_\mu = \frac{1}{2}\,(A^+_\mu + A^-_\mu)$$

$$B_\mu = \frac{k}{8\pi}\,(A^+_\mu - A^-_\mu)$$

The generating quantum functional is formally given by the path integral

$$Z_{BF,\Lambda}[M^3] = \int \mathcal{D}\mathbf{A}\mathcal{D}\mathbf{B}\,\exp\{i\,S_{BF,\Lambda}(\mathbf{A},\mathbf{B})\}$$

• There could be another coupling constant in front of the action, but it is just a numerical factor owing to the relation between Λ and k

Gauge–invariant (and Diff–invariant) quantum observables associated with oriented closed knotted curves C [1] embedded in M^3 $\leftrightarrow$ v.e.v. of Wilson loop operators

$$Z_{BF,\Lambda}[M^3,\ C] \;=\; \int \mathcal{D}\mathbf{A}\mathcal{D}\mathbf{B}\, \exp\left\{ i\, S_{BF,\Lambda}\left(\mathbf{A},\mathbf{B}\right)\right\} \mathrm{Tr}\,\mathrm{Hol}(\mathbf{A} \pm \Lambda\,\mathbf{B})$$

normalized by $(Z_{BF,\Lambda}[M^3])^{-1}$, where the holonomies

$$\mathrm{Hol}(\mathbf{A} \pm \Lambda\,\mathbf{B}) \;:=\; \exp\left\{ i \oint_C \left(A_\mu \pm \Lambda\, B_\mu\right) dz^\mu \right\}$$

are evaluated (up to path ordering) along the curve C parametrized by local coordinates z^μ

[1] knots and links, namely collections of disjoint interlaced knots

Not difficult to prove that at the formal level

$$Z_{BF,\Lambda}[M^3] = Z_{CS,k}[M^3]\ \overline{Z_{CS,k}[M^3]}$$

where $Z_{CS,k}[M^3]$ is the CS–Witten generating functional (a topological invariant) for the closed manifold M^3 and the complex conjugate Z refers to M^3 with the opposite orientation.
Observables:

$$Z_{BF,\Lambda}[M^3,\ C] =?\ Z_{CS,k}[M^3,\ C]\ \overline{Z_{CS,k}[M^3,\ C]}$$

where $Z_{CS,k}[M^3,\ C]$ represents a **polynomial invariants of knots (links)** of the Jones' type [V Jones 1985; E Witten 1989]

$$Z_{BF,\Lambda}[M^3] \leftleftarrows Z_{CS,k}[M^3]\,\overline{Z_{CS,k}[M^3]} \quad \text{again}$$

• The proof of interest here relies on an intermediate correspondence with the TV state sum

Preliminary remark: any closed smooth 3D manifold M^3 can be *presented* as the complement of a *framed knot* K (surgery knot) embedded in the 3-sphere S^3. The resulting 'quantum invariant' is a topological invariant [Reshetikhin–Turaev, 1991]) expressed as a weighted sum of colored Jones polynomials of the knot K and Z_{CS} (Witten 1989)$= Z_{RT}$

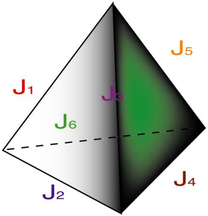
Double CS–BF with a Λ-term

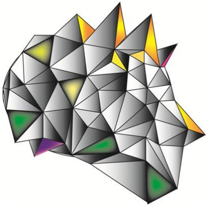
$$Z_{BF,\Lambda}[M^3,\, C] \;=\; Z_{CS,\,k}[M^3,\, C]\;\overline{Z_{CS,\,k}[M^3,\, C]}$$

• Knot invariants arising in the CS environment are **essentialy the same** as those found in double CS and thus in BF,$_\Lambda$ [A Cattaneo et al, J. Math. Phys. 36 (1995) 6137]

• Field–theoretical background: E Guadagnini *The link invariants of the Chern–Simons field theory*, 1993

Efficient quantum algorithms: S Garnerone, A M, M Rasetti, arXiv: quant–ph /0703037

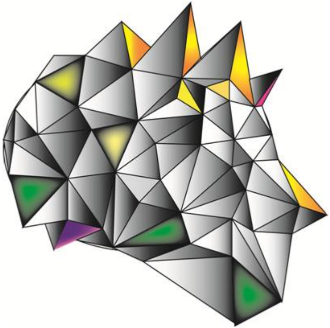




Any closed Piecewise–Linear (PL) manifold M^3 can be **presented** as a triangulation T^3 $(q; j_1, j_2, \ldots, j_N)$ made of colored tetrahedra glued along their triangular faces (Here colors are labels of *irreps* of $SU(2)_q$ at q a root of unity, $q = \exp(2\pi i/k)$)

The associated partition function or state sum is a topological (PL) invariant [Turaev–Viro, 1992], the regularized counterpart of Ponzano–Regge model for Euclidean 3D gravity

The Turaev–Viro state sum for a closed 3-manifold



Vertex $\to$ weight $\mathbf{w}(q)^{-2}$

Edge J $\to$
q–dimension $[2J+1]_q$

Tetrahedron $\to$
$q-6j$-symbol

$$\left\{ \begin{array}{ccc} J_1 & J_2 & J_3 \\ J_4 & J_5 & J_6 \end{array} \right\}_q$$

$$Z_{TV}[M^3, q] = \sum_{(T^3(J) \text{ of } M^3)} \prod_{vert} \mathbf{w}(q)^{-2N_0} \prod_{edges} [2J+1]_q \prod_{tetra} \{6j\}_q$$

N_0: number of vertices; sum over all colored triangulations

$$Z_{TV}[M^3, q] \Leftarrow Z_{CS,\,k}[M^3]\ \overline{Z_{CS,\,k}[M^3]} \quad (M^3 \text{oriented})$$

for each fixed value of $q = \exp(2\pi i/k)$

• Proof based on the *skein calculus* starting from the Reshetikhin–Turaev invariant quoted above [K Walker 1991, J Roberts, 1995]: from $M^3 \cup \overline{M}^3$ presented as the complement of a framed, colored knot, a colored triangulation is generated. *Invariance* relies on the improvement of suitable combinatorial moves, translated into algebraic identities for the q-6j symbols (pentagon and hexagon relations in the language of braided tensor categories) & 'Kirby' moves (...)

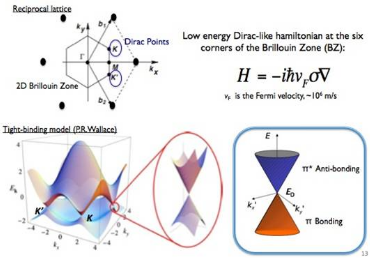
• Analogues of colored knot invariants (observables in quantum CS) can be defined, and the resulting *combinatorial expressions* agree with the path integral formulas, up to normalization

On the basis of the mutual correspondence with the double quantum CS (in the oriented case) the conclusion is:

$$Z_{BF,\Lambda}[M^3] = Z_{TV}[M^3, q]; \quad \Lambda = 4\pi/k$$

- Actually the result has been proven directly for the 3D Euclidean (gravitational + cosmological term) action in the first order formalism (with A the spin connection and B the dreibein)
- As discussed above the same correspondence holds for **observables**, in particular for $SU(2)$-colored knot invariants, for each value of q
- Extensions to colored triangulated 3-manifolds **with boundary** are crucial *e.g.* to characterize effective actions of topological nature in (2+1) lattice systems

At room temperature, near the Fermi points, electrons exhibit a relativistic behavior $\rightarrow$ (2+1)D massless Dirac equation



[A M, G Palumbo, *BF-theory in graphene: a route to topological quantum computing?* arXiv:1111.1593 & epl 2012]

$$i\gamma^\mu \partial_\mu \psi = 0$$

(Chiral rapresentation for the gamma matrices)
Bi–spinor (A,B refer to the two triangular sublattices):

$$\psi = \begin{pmatrix} \psi_A^+ \\ \psi_B^+ \\ \psi_A^- \\ \psi_B^- \end{pmatrix}$$

• One way for have a mass gap in this material is to add a chemical potential μ

• Topological defects in the honeycomb lattice generate vortices $\rightarrow$ coupling with $U(1) \times U(1)$ gauge fields a_μ and b_μ identified as the ordinary em field and the chiral gauge field [Ryu et al, 2009]

The low-energy action $S(a, b, \psi\overline{\psi})$ for coupled fermions in graphene is

$$S = -\beta \int d^3x \, \overline{\psi}^f (i\gamma^\mu \partial_\mu - \gamma^\mu a_\mu - \gamma_5 \gamma^\mu b_\mu - \mu\gamma_0)\psi_f$$

where β is a constant and the index $f = 1, 2$ takes into account the (real) spin degeneracy of particles

Further steps

- The replica method takes care of disorder–averaged physical quantities

$\Rightarrow$ the gauge fields are 1-forms in (the Lie algebra of) $U(N)xU(N)$

- Wick rotation on the background manifold
- Fermionic degrees of freedom integrated out
- Regularization (Pauli-Villars)
- Level-rank duality in gauge theories: $N \leftrightarrow k$ (CS coupling constant)

For β a positive integer the resulting action is a $U(2)_\beta \times U(2)_\beta$ Chern–Simons action, equivalent to the BF action

$$S_{BF,\Lambda} = \int d^3x \, \epsilon^{\lambda\mu\nu} \mathrm{Tr} \left(\mathbf{B}_\lambda \mathbf{F}_{\mu\nu} + \frac{\Lambda^2}{3} \mathbf{B}_\lambda \mathbf{B}_\mu \mathbf{B}_\nu \right) \tag{1}$$

where

$$\mathbf{A}_\mu = \mathbf{a}_\mu \, ; \quad \mathbf{B}_\mu = \frac{\beta}{4\pi} \mathbf{b}_\mu$$

- In bold: non–Abelian fields
- Natural 1–1 mapping with the gauge fields of the fermionic action, $\mathbf{a}_\mu \leftrightarrow$ connection and *shifted* $\mathbf{b}_\mu \leftrightarrow$ the chiral field reponsible for vortices
- Gauge invariant quantum observables $\leftrightarrow$ anyonic–type excitations (?)

Outlook (BF-TV environment)

- Beyond the monolayer flat graphene: effective topological actions for curved and multilayer graphene
- Beyond zero–temperature: perturbative expansions of quantum observables (?)
- **2D Boundaries & 1D embedded structures**: improving the comprehension of interconnections (BF with Λ term $\leftrightarrow$ TV) also in view of applications

$\Rightarrow$ Learn more on TV state sum models (next lectures), thus circumventing the path integral formalism of standard TQFT