Wave equation with energy-dependent potentials for confined systems

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Abstract
We study the properties of the wave equation for potentials depending on the energy with emphasis on confining potentials. In this case, for a linear energy dependence, the spectrum shows a saturation effect: the eigenvalues reach a finite limit as the quantum numbers increase. The harmonic oscillator and the linear potentials are studied as examples admitting analytical solutions. We apply such a model to the description of heavy quark systems. We first present a toy model based on the harmonic oscillator and show its ability to reproduce the experimental spectra of charmonium and bottomium. In more realistic calculations, use is made of the Cornell potential for the radial shape and an energy dependence more general than the linear assumption. Comparing the results with those of conventional potentials, we discuss to what extent energy-dependent potentials can bring new features in the description of heavy quark systems. Finally, we show that the energy dependence of the potential has a clear influence on the saturation of the spectrum.

1. Introduction

Wave equations with energy-dependent potentials are already familiar in physics. The relativistic description of a scalar particle in the presence of an external Coulomb field through the Klein–Gordon equation leads to a wave equation with an energy-dependent potential [1]. Even if the initial potential is not intrinsically energy dependent, the reduction to a wave equation introduces an effective potential depending on the energy [2, 3]. A similar situation occurs with the Dirac equation for a fermion in a scalar or vector potential, when reduced to the Pauli–Schrödinger equation [4, 5]. Two-particle relativistic equations such as the quasi-potential equations or constraint theory [6–8] lead to effective equations where the total energy (including mass terms) appears in the potential. In all these cases, the energy dependence is a relativistic effect. This dependence can be eliminated by employing a non-relativistic reduction and using the Schrödinger equation. Such a procedure gives rise to momentum-dependent terms.
In the present work, we follow the strategy of directly introducing energy-dependent potentials in the wave equation. It has the advantage of keeping these terms in a compact way. The result is a class of pseudo-Hermitian Hamiltonians, which could be used to describe nonlinear situations. The use of parametrized potentials is adequate to learn main features and general aspects from the wave equation and its solutions. However, in this way the link with a theory is lost. Consequently, working beyond a qualitative level requires a more fundamental approach. Note that in recent years potentials depending on the energy have been studied at a basic level [9–11], so that the handling of the corresponding wave equation became familiar.

The introduction of an energy dependence in the Schrödinger equation has several implications with respect to standard quantum mechanics. For example, the conservation of the norm asks for the modification of the scalar product [3, 12]. The energy-dependent potentials must fulfill certain conditions in order to result in a meaningful quantum theory [13]. For those potentials which are acceptable, the corresponding Schrödinger equation is equivalent to a Schrödinger equation including a non-local potential. The non-locality, however, is treated more efficiently under the form of an energy-dependent potential. A detailed analysis of these formal aspects is presented in [13] showing that the properties of a good quantum theory are well preserved.

The energy dependence of the potential brings interesting features to the spectrum, particularly in the case of confined systems. This was studied in [13] for the harmonic oscillator in the \( D = 1 \) dimensional space. In the first part of the present work, we shall recall these properties by considering two specific cases in the \( D = 3 \) dimensional space: the harmonic oscillator and the linear potential. For the sake of simplicity, we assume a spherical symmetry and a linear energy dependence. The latter assumption is justified if we consider a system of two heavy particles with equal masses. The energy dependence coming from the corresponding relativistic description is then linear [6–8].

A natural application of this model is offered by heavy quark systems. Their description has been a quite successful playground for potential models in the past [14]. These models have limited scopes, and more fundamental approaches have certainly more predictive power [15]. However, in spite of limitations, we believe potential models to underline some features of the physical situation in a simple and illustrative way. In the second part of the present work, we shall study the influence of an energy-dependent component in the potential on the calculated observables of charmonium and bottomonium. We first consider the harmonic oscillator as a toy model, showing that because of the energy dependence such a potential may yield a good fit to the experimental spectra. As a more realistic example, we take the radial shape of the Cornell potential [16] and discuss the effect of an energy dependence added to this potential. Calculations will involve the energy eigenvalues, the root-mean-square radii, the \( e^+e^- \) decay widths and the density of the \( s \) wavefunctions at the origin. The results will be compared to those obtained with the usual Cornell potential, as well as with the Buchmüller and Tye potential [17], known to give a fair description of both systems. Experimental data will also be quoted.

2. Confining potentials depending linearly on the energy

This section is devoted to two illustrative examples. The first one is the harmonic oscillator. We consider spinless particles. In the \( D = 3 \) dimensional space, the corresponding wave equation reads

\[
\left[ -\frac{\hbar^2}{2m} \nabla^2 + V(r, E_{n,\ell}) \right] \Psi_{n,\ell}(\vec{r}) = E_{n,\ell} \Psi_{n,\ell}(\vec{r}),
\]  

(1)
Wave equation with energy-dependent potentials for confined systems

where

\[ V(r, E_{n,\ell}) = \frac{1}{2} m \omega^2 (1 + \gamma E_{n,\ell}) r^2. \]  

(2)

The total Hamiltonian can be written as

\[ H = H_0 + V_1(r) H, \]  

(3)

where \( H_0 \) is the energy-independent part of \( H \). The first general property to check is the commutation relations of the angular momentum operator, namely

\[ [\ell_z, H] = [\ell_z, V_1 H]; \quad [\ell^2, H] = [\ell^2, V_1 H]. \]  

(4)

Since both operators act on angular variables, if \( V_1(r) \) is spherically symmetric, the commutators are zero. Thus, \( \ell \) and its projection on the \( z \)-axis are good quantum numbers. It allows the usual separation of radial and angular variables:

\[ \Psi_{n,\ell}(r, \theta, \phi) = \psi_{n,\ell}(r) Y_{\ell,m}(\theta, \phi). \]  

(5)

We recall that \( n - 1 \) denotes the number of nodes of the wavefunction. Note that the linearity of the energy dependence simplifies the calculation of commutation relations of observables with \( H \).

By setting \( \psi_{n,\ell}(r) = u_{n,\ell}(r)/r \), the reduced radial equation takes the form

\[ \left[ -\frac{d^2}{dr^2} + \frac{\ell(\ell + 1)}{r^2} + \Omega^2 (1 + \gamma E_{n,\ell}) r^2 \right] u_{n,\ell}(r) = \epsilon_{n,\ell} u_{n,\ell}. \]  

(6)

Here, \( \Omega = m \omega/\hbar \) and \( \epsilon_{n,\ell} = 2m E_{n,\ell}/\hbar^2 \). Referring to the work on the \( D = 1 \) harmonic oscillator [13], we conclude that this model is equivalent to the ordinary Schrödinger equation with a non-local potential, the non-locality depending only on the radial coordinate \( r \).

The general solution takes the usual form

\[ u_{n,\ell}(r) = C_{n,\ell} r^{\lambda_{n,\ell}} \exp \left( -\frac{\lambda_{n,\ell} r^2}{2} \right) _1 F_1 (-n + 1, \ell + 3/2; \lambda_{n,\ell} r^2), \]  

(7)

except that the decay coefficient \( \lambda \) is depending on the state. Here, \( _1 F_1 \) is the confluent hypergeometric function and \( C_{n,\ell} \) is the normalization factor.

The nonlinear character of the wave equation (1) is seen explicitly by writing the conditions connecting \( \lambda_{n,\ell} \) to the eigenvalues:

\[ \epsilon_{n,\ell} = \lambda_{n,\ell} (4n + 2\ell - 1) \]  

(8)

and

\[ \lambda_{n,\ell}^2 = \Omega^2 (1 + \Gamma \epsilon_{n,\ell}), \]  

(9)

where \( \Gamma = \gamma \hbar^2/2m \). It results in a second-order equation for the eigenvalues. The requirement of normalizable wavefunction imposes discarding the negative roots. The eigenvalues are then given by

\[ E_{n,\ell} = \hbar^2 \omega \left( \frac{4n + 2\ell - 1}{8} \gamma + \hbar \omega \frac{(4n + 2\ell - 1)}{2} \sqrt{1 + \frac{\hbar^2 \omega^2 (4n + 2\ell - 1)^2}{16}} \right)^{\gamma^2}. \]  

(10)

As shown in [13], a coherent model is met only for \( \gamma < 0 \).

An example of the spectrum is given in figure 1, for \( \gamma = -0.25 \). The calculation has been done for \( \hbar = m = \omega = 1 \). The results show clearly two different effects compared to the ordinary harmonic oscillator. First, the ground-state energy \( E_{0,0} \) is lowered with respect to the rest of the spectrum. Second, instead of growing infinitely, the spectrum exhibits saturation.
As the quantum numbers increase, the energy reaches a limit. This limit is easily deduced from equation (10):

$$\lim_{n,\ell \to \infty} E_{n,\ell} = \frac{1}{|\gamma|}. \tag{11}$$

Consequently, the separation between the states tends to zero and the density of the states becomes infinite.

The second studied example is the linear potential:

$$V(r, E_{n,\ell}) = m\sigma (1 + \gamma E_{n,\ell})r. \tag{12}$$

For the same reason as for the harmonic oscillator, only a negative $\gamma$ yields a consistent model. It admits analytical solutions only for the s states. It starts with the transformation

$$\frac{2m}{\hbar^2} \sigma (1 + \gamma E_{n,0}) = \frac{1}{\lambda_{n,0}^2} \tag{13}$$

together with

$$z = r - \epsilon_{n,0}. \tag{14}$$

It transforms equation (1) into the differential equation of the Airy function. The quantization condition links the $n$th eigenvalue to the $n$th zeros of the Airy function $a_n$:

$$-a_n = \lambda_{n,0}^2 \frac{2m}{\hbar^2} E_{n,0}. \tag{15}$$

The s-state spectrum is then given by a cubic equation:

$$(-a_n)^{3/2} \sqrt{\frac{m}{\hbar^2}} \sigma (1 + \gamma E_{n,0}) = E_{n,0}^{3/2}. \tag{16}$$

As it can be checked, this equation admits a single real positive root for $\gamma < 0$. The asymptotic behaviour is obtained by means of the asymptotic expression for $a_n$. It reads

$$a_n \approx -\left(\frac{3\pi}{2} n\right)^{2/3}; \quad n \geq 10. \tag{17}$$
By inserting this value into equation (16) and letting \( n \to \infty \), we get

\[
\lim_{n \to \infty} E_{n,0} = \frac{1}{|\gamma|}.
\]

(18)

For \( \ell \neq 0 \) states, the wave equation has to be solved numerically. As for the harmonic oscillator, the eigenvalues are displayed in figure 1 for \( E_{n,0} \) and \( E_{1,\ell} \).

3. Energy dependence and saturation

The energy dependence of the potential should be determined by the underlying theory, depending on the specific system considered. General requirements such as relativistic invariance and the uniqueness of the energy eigenvalue for a given state are not sufficient for a unique determination of the energy dependence. In the present work, studying its effect at the level of the wave equation, we first focused our attention on a linear dependence. In this section, we consider a more general form suggested by the Pauli–Schrödinger equation [4, 5]:

\[
V(r, E_{n,\ell}) = V_0(r) \left[ 1 + \frac{E_{n,\ell}}{2m} \right]^\gamma,
\]

(19)

with \( \gamma \) being an adjustable parameter.

Going beyond the linear energy dependence raises two questions.

The first point concerns the uniqueness of the eigenvalue solution. Whereas it is allowed to have several states with the same energy (degeneracy), it is illegal to have several energies for the same state. In other words, as quoted in the preceding section, we have to ensure that in spite of the nonlinear equation determining the energy, only a single solution is acceptable.

This task is difficult in the case of the above potential. However, we can get a hint by considering the following remark. From the harmonic and linear potentials, we have learned that the equation to be solved takes the form

\[
E_{p_{n,\ell}}^p = A \left( 1 + \frac{E_{n,\ell}}{2m} \right)^\gamma,
\]

(20)

where \( A > 0 \). For the positive sector \( E_{n,\ell} > 0 \) and negative \( \gamma \), it is easy to show that the right-hand side of equation (20) is a continuously decreasing function of \( E_{n,\ell} \). Consequently, for any positive power \( p > 0 \), this equation has a single solution on \( E_{n,\ell} \in [0, \infty] \). The argument is not necessarily valid for an arbitrary radial shape. Nevertheless, we conjecture that it is indeed satisfied for any confining potential provided that the whole spectrum is positive definite.

The second point concerns the behaviour of the spectrum for large quantum numbers. For both radial shapes studied in section 2, the spectrum is reaching a finite upper limit as \( n \) or \( \ell \) increase to infinity. This property is linked to the linear energy dependence which was used. Actually, the saturation was also found in [13] for a square-root dependence, for the harmonic oscillator in \( D = 1 \) dimensional space, namely for \( V(r) = \omega^2 r^2 (1 + \gamma \sqrt{E_n}) \). However, this is not a general property.

To show the influence of the energy dependence on the spectrum for large quantum numbers, we take the following examples. Setting for simplicity \( m = \hbar = 1 \) and taking \( \gamma = -1 \), we use expression (19) together with the harmonic oscillator and the linear radial shapes. In both cases, the calculation of the s-state eigenvalues is analytic.

For the harmonic oscillator with \( \omega = 1 \), the s-state eigenvalues are given by

\[
E_n = \lambda_n (4n - 1); \quad \lambda_n^2 = \frac{1}{1 + E_n/2},
\]

(21)
Solving this system gives the following asymptotic behaviour:

$$\lim_{n \to \infty} E_n \approx 2^{5/3}n^{2/3}. \quad (22)$$

The spectrum does not saturate, although the energy increase is sensibly reduced with respect to the ordinary harmonic oscillator.

In the case of the linear potential, setting $\sigma = 1$, the energy of the $s$ state for $n > 10$ is given by the equivalent to equation (21), namely solutions of

$$E_n^{2/3} = \frac{3\pi}{2\sqrt{2}} n \left(1 + \frac{\ell^2}{2n}\right). \quad (23)$$

The asymptotic expression reads

$$\lim_{n \to \infty} E_n \approx \left[\frac{2\sqrt{2}}{3\pi n}\right]^{2/5}. \quad (24)$$

As in the previous example, the saturation is lost but the rate of increase is clearly reduced.

4. Application to heavy quarks systems

As stated in the introduction, heavy quark systems constitute a natural domain of application of the wave equation with energy-dependent potentials. In this section, we shall confront the model with charmonium and bottomium data, and compare our results with those obtained with conventional potentials. We shall limit the discussion to the triplet states.

4.1. A toy model

The harmonic oscillator is not a good starting point to describe $q\bar{q}$ hadrons. Experimentally, the $1d$ state is above the $2s$ state, and the energy dependence does not remove their degeneracy (see figure 1). On the other hand, with a small modification, it is possible to account for this situation with a potential still admitting analytical solutions. Since it is often useful to have a simple model at our disposal, we keep a linear energy dependence and start with the following example:

$$V(r, E_{n,\ell}) = \left[\frac{m\omega^2}{2}r^2 + D_0\ell^2\right](1 + \gamma E_{n,\ell}). \quad (25)$$

Here, $m$ is the reduced mass. The $\ell^2$ term is known to simulate a square well shape when associated with the harmonic oscillator. The free parameters $\omega$, $D_0$ and $\gamma$ are adjusted to the energies of the $1p$, $2s$ and $3s$ states with respect to $E_{1s}$, taken from the experimental data, for $c\bar{c}$ and $b\bar{b}$ [28, 29] separately. The quark mass is connected to the physical mass by

$$M(q\bar{q}) = 2m_q + E_{1s}. \quad (26)$$

Setting

$$G_1 = 1 - D_0\ell(\ell + 1)\gamma \quad \text{and} \quad G_2 = \frac{\hbar\omega[(4n + 2\ell - 1)]}{2},$$

the eigenvalues are given by

$$E_{n,\ell} = \frac{1}{2G_1^2} \left[G_1^2\gamma^2 - 2D_0\ell(\ell + 1)G_1 + 2G_2\sqrt{G_1^2 + \frac{G_2^2\gamma^2}{4}}\right]. \quad (27)$$
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Table 1. The $c\bar{c}$ and $b\bar{b}$ properties described by the harmonic oscillator potential (25): the results display the energy differences (MeV) with respect to the ground state, the $e^+e^-$ decay widths (keV) and the root-mean-square radius (fm). Experimental data are given for comparison, when available [28, 29].

<table>
<thead>
<tr>
<th>Spectrum</th>
<th>Experimental</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Charmionium</td>
</tr>
<tr>
<td>1p</td>
<td>397</td>
</tr>
<tr>
<td>2s</td>
<td>589</td>
</tr>
<tr>
<td>1d</td>
<td>713</td>
</tr>
<tr>
<td>2p</td>
<td>819</td>
</tr>
<tr>
<td>3s</td>
<td>941</td>
</tr>
<tr>
<td>4s</td>
<td>1153</td>
</tr>
</tbody>
</table>

| W.F. properties | | |
|-----------------|-----------------|
| $\Gamma_{1s}/\Gamma_{2s}$ | 2.65/5.3 | 0.47/1.32 |
| $\langle r^2 \rangle_{1s}$ | 0.490/0.26 |

| Parameters | | |
|------------|-----------------|
| $\omega$ (fm$^{-1}$) | 2.75 | 2.65 |
| $D_0$ (MeV) | 85.3 | 153.0 |
| $\gamma \times 10^{-4}$ (MeV$^{-1}$) | -4.336 | -4.558 |
| $m$ (GeV) | 1.207 | 4.401 |

It is interesting to test the properties of the wavefunctions. The $e^+e^-$ decay widths of the $s$ states provide us with such a possibility. In the absence of the radiative and relativistic corrections [18–20], they are given by the van Royen–Weisskopf formula [21]:

$$\Gamma_{e^+e^-}(ns) = \frac{16\pi e_q^2\alpha^2}{M^2(q\bar{q})} |\psi_{n,0}(0)|^2.$$  \hspace{1cm} (28)

Here, $e_q$ is the quark charge and $\alpha$ is the fine structure constant. We have verified that the same formula holds when using energy-dependent potentials at the lowest order.

In this respect, it is to be noted that the van Royen–Weisskopf formula should be multiplied by a factor $(1-16\alpha_s/3\pi)$ corresponding to the (first-order) radiative corrections [17, 27]. In [22, 23] it is shown that after the inclusion of the relativistic corrections, potential models having $|R_{1s}(0)|^2 = 0.8–1.0$ GeV$^3$ for $c\bar{c}$ and 6.0–8.0 GeV$^3$ for the $b\bar{b}$ are able to reproduce the $e^+e^-$ decay widths. This point should be reconsidered in the framework of energy-dependent potentials. Since we compare different approaches, we shall be satisfied with the zero-order estimate equation (28).

The results are summarized in table 1. Whereas the spectra are well reproduced, the $e^+e^-$ decay widths are too small by a factor 2–3. It means that the potential does not concentrate enough $s$-state wavefunctions at the origin.

It is interesting to note that the parameters are nearly the same for $c\bar{c}$ and $b\bar{b}$, except for $D_0$. Note also that the energy dependence is essential to reproduce the spectrum with the harmonic oscillator radial shape. This is particularly clear from the $s$-state energies, which in the usual case grow linearly with $n$ and are independent on the $\ell^2$ term.

It is tempting to play a similar game with the linear potential. However, fitting the spectra requires again an $\ell^2$ term, though of smaller strength than in the harmonic oscillator case. On the other hand, the linear potential produces higher $s$-state wavefunctions at the origin.
Table 2. $c\bar{c}$ properties: results obtained by adding energy dependence to the Cornell potential equation (29). For the sake of comparison, we quote the results given by the Buchmüller and Tye (BT) [17] and the usual Cornell [16] potentials. The values listed correspond to the energies (MeV) with respect to the ground state, to the $e^+e^-$ decay widths (keV), the root-mean-square radius (fm) and the density at the origin (GeV$^3$). The parameters used are $\kappa = 98$ MeV fm, $\sigma = 1190$ MeV fm$^{-1}$, $\gamma = -0.8$ and $m = 1600$ MeV. Experimental data are also given [28, 29].

<table>
<thead>
<tr>
<th>State</th>
<th>This work</th>
<th>BT</th>
<th>Cornell</th>
<th>Experimental</th>
</tr>
</thead>
<tbody>
<tr>
<td>1p</td>
<td>402</td>
<td>424</td>
<td>428</td>
<td>397</td>
</tr>
<tr>
<td>2s</td>
<td>572</td>
<td>601</td>
<td>590</td>
<td>590</td>
</tr>
<tr>
<td>1d</td>
<td>664</td>
<td>716</td>
<td>713</td>
<td>706</td>
</tr>
<tr>
<td>2p</td>
<td>814</td>
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</tr>
<tr>
<td>4s</td>
<td>1254</td>
<td>1376</td>
<td>1374</td>
<td>1063</td>
</tr>
<tr>
<td>$\Gamma_{1s}$</td>
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<td>8.0</td>
<td>14.33</td>
<td>5.4(0.21)</td>
</tr>
<tr>
<td>$\Gamma_{2s}/\Gamma_{1s}$</td>
<td>2.55</td>
<td>2.17</td>
<td>2.22</td>
<td>2.52</td>
</tr>
<tr>
<td>$\langle r^2 \rangle_{1s}^{1/2}$</td>
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<td>0.42</td>
<td>0.47</td>
<td></td>
</tr>
<tr>
<td>$</td>
<td>R_{1s}(0)</td>
<td>^2$</td>
<td>1.01</td>
<td>0.81</td>
</tr>
</tbody>
</table>

We have verified that the $e^+e^-$ decay widths are enlarged by 50% compared to the results displayed in table 1. It is not sufficient to meet the experimental values. Consequently, this case is not worth investigating beyond the qualitative arguments just given.

4.2. A more realistic case

Many potentials have been proposed for the description of heavy quarks systems [15, 16, 23–26]. Some of them are based on arguments drawn from spectroscopic data, while others invoke constraints arising from fundamental aspects of QCD.

Our purpose being to underline the new features brought by the introduction of an energy-dependent potential in the wave equation, we shall restrict our investigation to the Cornell potential taken as a reasonable example. It consists of two components which are generally well accepted: a $1/r$ term corresponding to the one-gluon exchange and a linear confinement. As far as the energy dependence is concerned, we choose a more general form than for the previous examples. Also the connection to relativistic effects is made more transparent by taking an explicit $1/m$-dependence of the coefficient. Consequently, the potential to be studied is given by

$$V(r, E_{n,\ell}) = \left(-\frac{\kappa}{r} + \sigma r\right) \left(1 + \frac{E_{n,\ell}}{2m}\right)^\gamma.$$  (29)

The results of our calculations are displayed in tables 2 and 3, as well as in figures 2 and 3, for $c\bar{c}$ and $b\bar{b}$, respectively. The fits have been achieved simultaneously on both systems. Consequently, the two parameters $\sigma$ and $\gamma$ are the same. The parameter $\kappa$ scales with the mass, in a ratio of 1.26, to be compared to 1.28 according to the known values of $\alpha_s$ [29].

From the displayed results, we note the large effect of the energy dependence on the energies of the 3s and 4s states, as well as on the $\Gamma_{1s}$, compared to the result of the Cornell potential. In the $b\bar{b}$ case, adding the energy dependence to the Cornell potential brings the results in close agreement with those of the BT potential, and thus close to the experimental data (see figure 3).

The fit to the experimental data is less satisfactory for the $c\bar{c}$ system, as illustrated on energy spectra plotted in figure 2. Relaxing the constraint of keeping fixed $\sigma$ and $\gamma$ would
Figure 2. $c\bar{c}$ spectrum calculated using the energy-dependent potential of the form defined in equation (29) (this work). The results yielded by the Buchmüller and Tye (BT) [17] and Cornell [16] potentials and the experimental data (dashed lines) are presented for comparison. The plotted values correspond to energies with respect to the ground state. The parameters used are given in Table 2.

Table 3. $b\bar{b}$ properties: same as Table 2, but here the results correspond to $\kappa = 78 \text{ MeV fm}$, $\sigma = 1190 \text{ MeV fm}^{-1}$, $\gamma = -0.8$ and $m = 5200 \text{ MeV}$.

<table>
<thead>
<tr>
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<th>This work</th>
<th>BT</th>
<th>Cornell</th>
<th>Experimental</th>
</tr>
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<tr>
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<tr>
<td>$\Gamma_{1s}$</td>
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<td>3.66</td>
<td>1.31(4.03)</td>
</tr>
<tr>
<td>$\Gamma_{1s}/\Gamma_{2s}$</td>
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<td>2.2</td>
<td>2.17</td>
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<tr>
<td>$\langle r^2 \rangle_{1s}$</td>
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<td>0.23</td>
<td>0.20</td>
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<tr>
<td>$</td>
<td>R_{1s}(0)</td>
<td>^2$</td>
<td>8.61</td>
<td>6.48</td>
</tr>
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</table>

improve the agreement. However, this is not our goal at the present stage. The real question is to know whether the energy dependence of the potential is a key factor at a phenomenological level. The relativistic nature of this effect argues in favour of its introduction in the wave equation. On the other hand, the fit to the low energy part of the spectra is not sufficient to establish its necessity. This is exemplified by the good agreement obtained with the BT potential for the bottomium.
5. Conclusions

The present work is devoted to the study of the wave equation with potentials depending on the energy. Special emphasis is given to confining potential. For models leading to a consistent quantum mechanics, the spectrum shows two major changes: the lowering of the $n = 1$ states (no node in the radial wavefunction) with respect to other states and a saturation effect. As the quantum numbers increase, the eigenvalues reach a limit and the density of states becomes infinite.

These results have been obtained with a schematic model, involving simple radial shapes for the potential and a linear energy dependence. This choice allows analytical solutions of the wave equation. It is quite efficient at the qualitative level, illustrating the main features of the problem.

Applying the model to heavy quark systems, it is surprising at first glance that a model based on a harmonic oscillator shape yields spectra in good agreement with experiment. The relevant point is provided us by the energies of the s states. They cannot be given by the harmonic oscillator without the energy dependence. Note also that the s-state spectrum is not very sensitive to the radial shape of the potential, once the energy dependence is introduced.

To go beyond qualitative aspects, we have studied a more realistic case. The radial shape is taken from the Cornell potential and the energy dependence is the one expected from relativistic effects. The results obtained with this model are satisfactory, when compared to the Buchmüller and Tye results, as well as to experiments. They improve the original Cornell potential, both in the spectra and in the zero-order estimate of $\Gamma_{1s}$. 

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Figure 3. Same as figure 2, but for $b\bar{b}$ spectrum. The parameters used are given in table 3.
The remaining question is to know to what extent the energy dependence is required by the experimental data. The present study clearly shows that the low energy spectrum together with the $e^+e^-$ decay widths are not sufficient to decide. This is exemplified by the excellent agreement resulting from the Buchmüller and Tye potential for $b\bar{b}$. The identification of higher states or the evidence of the saturation effect would be the key to the answer.

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References


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